ON $f$-DERIVATIONS OF BCI-ALGEBRAS

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The notion of left-right (resp., right-left) $f$-derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular $f$-derivation, we give characterizations of a $p$-semisimple BCI-algebra.

1. Introduction and preliminaries

In the theory of rings and near-rings, the properties of derivations are an important topic to study, see [2, 3, 7, 10]. In [6], Jun and Xin applied the notions in rings and near-rings theory to BCI-algebras, and obtained some related properties. In this paper, the notion of left-right (resp., right-left) $f$-derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular $f$-derivation, we give characterizations of a $p$-semisimple BCI-algebra.

By a BCI-algebra we mean an algebra $(X; *, 0)$ of type (2,0) satisfying the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$;
(II) $(x * (x * y)) * y = 0$;
(III) $x * x = 0$;
(IV) $x * y = 0$ and $y * x = 0$ imply that $x = y$;
for all $x, y, z \in X$.

In any BCI-algebra $X$, one can define a partial order “$\leq$” by putting $x \leq y$ if and only if $x * y = 0$.

A subset $S$ of a BCI-algebra $X$ is called subalgebra of $X$ if $x * y \in S$ for all $x, y \in S$. A subset $I$ of a BCI-algebra $X$ is called an ideal of $X$ if it satisfies (i) $0 \in I$; (ii) $x * y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

A mapping $f$ of a BCI-algebra $X$ into itself is called an endomorphism of $X$ if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that $f(0) = 0$. Especially, $f$ is monic if for any $x, y \in X$, $f(x) = f(y)$ implies that $x = y$.

A BCI-algebra $X$ has the following properties:

1. $x * 0 = x$;
2. $(x * y) * z = (x * z) * y$;
On $f$-derivations of BCI-algebras

(3) $x \leq y$ implies that $x \ast z \leq y \ast z$ and $z \ast y \leq z \ast x$;
(4) $x \ast (x \ast (x \ast y)) = x \ast y$;
(5) $(x \ast z) \ast (y \ast z) \leq x \ast y$;
(6) $0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y)$;
(7) $x \ast 0 = 0$ implies that $x = 0$.

For a BCI-algebra $X$, denote by $X_+$ (resp., $G(X)$) the BCK-part (resp., the BCI-G part) of $X$, that is, $X_+ = \{x \in X \mid 0 \leq x\}$ (resp., $G(X) = \{x \in X \mid 0 \ast x = x\}$). Note that $G(X) \cap X_+ = \{0\}$. If $X_+ = \{0\}$, then $X$ is called a $p$-semisimple BCI-algebra.

In a $p$-semisimple BCI-algebra $X$, the following hold:

(8) $(x \ast z) \ast (y \ast z) = x \ast y$;
(9) $0 \ast (0 \ast x) = x$;
(10) $x \ast (0 \ast y) = y \ast (0 \ast x)$;
(11) $x \ast y = 0$ implies that $x = y$;
(12) $x \ast a = x \ast b$ implies that $a = b$;
(13) $a \ast x = b \ast x$ implies that $a = b$;
(14) $a \ast (a \ast x) = x$.

Let $X$ be a $p$-semisimple BCI-algebra. We define addition “+” as $x + y = x \ast (0 \ast y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity $0$ and $x - y = x \ast y$. Conversely, let $(X, +)$ be an abelian group with identity $0$ and let $x \ast y = x - y$. Then $X$ is a $p$-semisimple BCI-algebra and $x + y = x \ast (0 \ast y)$ for all $x, y \in X$ (see [5]).

For a BCI-algebra $X$, we denote $x \land y = y \ast (y \ast x)$, in particular, $0 \ast (0 \ast x) = a_x$, and $L_p(X) = \{a \in X \mid x \ast a = 0 \Rightarrow x = a\}$ for any $x \in X$. We call the elements of $L_p(X)$ the $p$-atoms of $X$. For any $a \in X$, let $V(a) = \{x \in X \mid a \ast x = 0\}$, which is called the branch of $X$ with respect to $a$. It follows that $x \land y \in V(a \ast b)$ whenever $x \in V(a)$ and $y \in V(b)$ for all $x, y \in X$ and $a, b \in L_p(X)$. Note that $L_p(X) = \{x \in X \mid a_x = x\}$, which is the $p$-semisimple part of $X$, and $X$ is a $p$-semisimple BCI-algebra if and only if $L_p(X) = X$ (see [6]). Note also that $a_x \in L_p(X)$, that is, $0 \ast (0 \ast a_x) = a_x$, which implies that $a_x \ast y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subseteq L_p(X), x \land (x \ast a) = a$, and $a \ast x \in L_p(X)$ for all $a \in L_p(X)$ and $x \in X$. For more details, refer to [1, 8, 11].

**Definition 1.1** [9]. A BCI-algebra $X$ is said to be **commutative** if $x = x \land y$ whenever $x \leq y$ for all $x, y \in X$.

**Definition 1.2** [4]. A BCI-algebra $X$ is said to be **branchwise commutative** if $x \land y = y \land x$ for all $x, y \in V(a)$ and all $a \in L_p(X)$.

**Lemma 1.3** [6]. A BCI-algebra $X$ is commutative if and only if it is branchwise commutative.

**Definition 1.4** [6]. Let $X$ be a BCI-algebra. By a **left-right derivation** (briefly, $(l, r)$-derivation) of $X$, a self-map $d$ of $X$ satisfying the identity $d(x \ast y) = (d(x) \ast y) \land (x \ast d(y))$ for all $x, y \in X$ is meant. If $d$ satisfies the identity $d(x \ast y) = (x \ast d(y)) \land (d(x) \ast y)$ for all $x, y \in X$, then it is said that $d$ is a **right-left derivation** (briefly, $(r, l)$-derivation) of $X$. Moreover, if $d$ is both an $(r, l)$- and an $(l, r)$-derivation, it is said that $d$ is a **derivation**.

2. $f$-derivations

In what follows, let $f$ be an endomorphism of $X$ unless otherwise specified.
Definition 2.1. Let $X$ be a BCI-algebra. By a left-right $f$-derivation (briefly, $(l,r)$-$f$-derivation) of $X$, a self-map $d_f$ of $X$ satisfying the identity $d_f(x \ast y) = (d_f(x) \ast f(y)) \land (f(x) \ast d_f(y))$ for all $x, y \in X$ is meant, where $f$ is an endomorphism of $X$. If $d_f$ satisfies the identity $d_f(x \ast y) = (f(x) \ast d_f(y)) \land (d_f(x) \ast f(y))$ for all $x, y \in X$, then it is said that $d_f$ is a right-left $f$-derivation (briefly, $(r,l)$-$f$-derivation) of $X$. Moreover, if $d_f$ is both an $(r,l)$- and an $(l,r)$-$f$-derivation, it is said that $d_f$ is an $f$-derivation.

Example 2.2. Let $X = \{0,1,2,3,4,5\}$ be a BCI-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>0</td>
</tr>
</tbody>
</table>

Define a map $d_f : X \to X$ by

$$d_f(x) = \begin{cases} 2 & \text{if } x = 0, 1, \\ 0 & \text{otherwise}, \end{cases} \quad (2.1)$$

and define an endomorphism $f$ of $X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1, \\ 2 & \text{otherwise}. \end{cases} \quad (2.2)$$

Then it is easily checked that $d_f$ is both derivation and $f$-derivation of $X$.

Example 2.3. Let $X$ be a BCI-algebra as in Example 2.2. Define a map $d_f : X \to X$ by

$$d_f(x) = \begin{cases} 2 & \text{if } x = 0, 1, \\ 0 & \text{otherwise}. \end{cases} \quad (2.3)$$

Then it is easily checked that $d_f$ is a derivation of $X$.

Define an endomorphism $f$ of $X$ by

$$f(x) = 0, \quad \forall x \in X. \quad (2.4)$$

Then $d_f$ is not an $f$-derivation of $X$ since

$$d_f(2 \ast 3) = d_f(0) = 2, \quad (2.5)$$

but

$$(d_f(2) \ast f(3)) \land (f(2) \ast d_f(3)) = (0 \ast 0) \land (0 \ast 0) = 0 \land 0 = 0, \quad (2.6)$$

and thus $d_f(2 \ast 3) \neq (d_f(2) \ast f(3)) \land (f(2) \ast d_f(3))$. 

From Example 2.3, we know that there is a derivation of $X$ which is not an $f$-derivation of $X$.

**Example 2.5.** Let $X = \{0,1,2,3,4,5\}$ be a BCI-algebra with the following Cayley table:

<table>
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<th>0</th>
<th>1</th>
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<td>4</td>
<td>1</td>
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<td>0</td>
</tr>
</tbody>
</table>

Define a map $d_f : X \rightarrow X$ by

$$d_f(x) = \begin{cases} 
0 & \text{if } x = 0,1, \\
2 & \text{if } x = 2,4, \\
3 & \text{if } x = 3,5,
\end{cases} \quad (2.7)$$

and define an endomorphism $f$ of $X$ by

$$f(x) = \begin{cases} 
0 & \text{if } x = 0,1, \\
2 & \text{if } x = 2,4, \\
3 & \text{if } x = 3,5.
\end{cases} \quad (2.8)$$

Then it is easily checked that $d_f$ is both derivation and $f$-derivation of $X$.

**Example 2.6.** Let $X$ be a BCI-algebra as in Example 2.5. Define a map $d_f : X \rightarrow X$ by

$$d_f(x) = \begin{cases} 
0 & \text{if } x = 0,1, \\
2 & \text{if } x = 2,4, \\
3 & \text{if } x = 3,5.
\end{cases} \quad (2.9)$$

Then it is easily checked that $d_f$ is a derivation of $X$.

Define an endomorphism $f$ of $X$ by

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 3, \quad f(3) = 2, \quad f(4) = 5, \quad f(5) = 4. \quad (2.10)$$

Then $d_f$ is not an $f$-derivation of $X$ since

$$d_f(2 * 3) = d_f(3) = 3, \quad (2.11)$$

but

$$(d_f(2) * f(3)) \land (f(2) * d_f(3)) = (2 * 2) \land (3 * 3) = 0 \land 0 = 0, \quad (2.12)$$

and thus $d_f(2 * 3) \neq (d_f(2) * f(3)) \land (f(2) * d_f(3))$. 

Example 2.7. Let $X$ be a BCI-algebra as in Example 2.5. Define a map $d_f : X \rightarrow X$ by

$$
d_f(0) = 0, \quad d_f(1) = 1, \quad d_f(2) = 3, \quad d_f(3) = 2, \quad d_f(4) = 5, \quad d_f(5) = 4. \quad (2.13)
$$

Then $d_f$ is not a derivation of $X$ since

$$
d_f(2 \ast 3) = d_f(3) = 2, \quad \text{but} \quad (d_f(2) \ast 3) \land (2 \ast d_f(3)) = (3 \ast 3) \land (2 \ast 2) = 0 \land 0 = 0, \quad (2.15)
$$

and thus $d_f(2 \ast 3) \neq (d_f(2) \ast 3) \land (2 \ast d_f(3))$.

Define an endomorphism $f$ of $X$ by

$$
f(0) = 0, \quad f(1) = 1, \quad f(2) = 3, \quad f(3) = 2, \quad f(4) = 5, \quad f(5) = 4. \quad (2.16)
$$

Then it is easily checked that $d_f$ is an $f$-derivation of $X$.

Remark 2.8. From Example 2.7, we know that there is an $f$-derivation of $X$ which is not a derivation of $X$.

For convenience, we denote $f_x = 0 \ast (0 \ast f(x))$ for all $x \in X$. Note that $f_x \in L_p(X)$.

Theorem 2.9. Let $d_f$ be a self-map of a BCI-algebra $X$ defined by $d_f(x) = f_x$ for all $x \in X$. Then $d_f$ is an $(l, r)$-$f$-derivation of $X$. Moreover, if $X$ is commutative, then $d_f$ is an $(r, l)$-$f$-derivation of $X$.

Proof. Let $x, y \in X$.

Since

$$
0 \ast (0 \ast (f_x \ast f(y))) = 0 \ast (0 \ast ((0 \ast (0 \ast f(x))) \ast f(y))) \\
= 0 \ast (0 \ast ((0 \ast f(y)) \ast (0 \ast f(x)))) \\
= 0 \ast (0 \ast (0 \ast f(y \ast x))) = 0 \ast f(y \ast x) \\
= 0 \ast (f(y) \ast f(x)) = (0 \ast f(y)) \ast (0 \ast f(x)) \\
= (0 \ast (0 \ast f(x))) \ast f(y) = f_x \ast f(y),
$$

we have $f_x \ast f(y) \in L_p(X)$, and thus

$$
f_x \ast f(y) = (f(x) \ast f_y) \ast ((f(x) \ast f_y) \ast (f_x \ast f(y))). \quad (2.18)
$$

It follows that

$$
d_f(x \ast y) = f_{x \ast y} = 0 \ast (0 \ast f(x \ast y)) = 0 \ast (0 \ast (f(x) \ast f(y))) \\
= (0 \ast (0 \ast f(x))) \ast (0 \ast (0 \ast f(y))) = f_x \ast f_y \\
= (0 \ast (0 \ast f_x)) \ast (0 \ast (0 \ast f(y))) = 0 \ast (0 \ast (f_x \ast f(y))) \\
= f_x \ast f(y) = (f(x) \ast f_y) \ast ((f(x) \ast f_y) \ast (f_x \ast f(y))) \\
= (f_x \ast f(y)) \land (f(x) \ast f_y) = (d_f(x) \ast f(y)) \land (f(x) \ast d_f(y)), \quad (2.19)
$$
and so \( df \) is an \((l,r)\)-\(f\)-derivation of \( X \). Now, assume that \( X \) is commutative. Using Lemma 1.3, it is sufficient to show that \( df(x) \ast f(y) \) and \( f(x) \ast df(y) \) belong to the same branch for all \( x, y \in X \), we have

\[
df(x) \ast f(y) = f_x \ast f(y) = 0 \ast (0 \ast (f_x \ast f(y)))
= (0 \ast (0 \ast f_x)) \ast (0 \ast (0 \ast f(y)))
= f_x \ast f_y \in V(f_x \ast f_y),
\]

and so \( f_x \ast f_y = (0 \ast (0 \ast f(x))) \ast (0 \ast (0 \ast f(y))) = 0 \ast (0 \ast (f(x) \ast f_y)) = 0 \ast (0 \ast f(x) \ast df(y)) \leq f(x) \ast df(y) \), which implies that \( f(x) \ast df(y) \in V(f_x \ast f_y) \). Hence, \( df(x) \ast f(y) \) and \( f(x) \ast df(y) \) belong to the same branch, and so

\[
df(x \ast y) = (df(x) \ast f(y)) \land (f(x) \ast df(y))
= (f(x) \ast df(y)) \land (df(x) \ast f(y)).
\]

This completes the proof. \( \square \)

**Proposition 2.10.** Let \( df \) be a self-map of a BCI-algebra \( X \). Then the following hold.

(i) If \( df \) is an \((l,r)\)-\(f\)-derivation of \( X \), then \( df(x) = df(x) \land f(x) \) for all \( x \in X \).

(ii) If \( df \) is an \((r,l)\)-\(f\)-derivation of \( X \), then \( df(x) = f(x) \land df(x) \) for all \( x \in X \) if and only if \( df(0) = 0 \).

**Proof.**  
(i) Let \( df \) be an \((l,r)\)-\(f\)-derivation of \( X \). Then,

\[
df(x) = df(x \ast 0) = (df(x) \ast f(0)) \land (f(x) \ast df(0))
= (df(x) \ast 0) \land (f(x) \ast df(0)) = df(x) \land (f(x) \ast df(0))
= (f(x) \ast df(0)) \ast ((f(x) \ast df(0)) \ast df(x))
= (f(x) \ast df(0)) \ast (f(x) \ast df(x) \ast df(x))
\leq f(x) \ast f(x) \ast df(x) = df(x) \land f(x).
\]

But \( df(x) \land f(x) \leq df(x) \) is trivial and so (i) holds.

(ii) Let \( df \) be an \((r,l)\)-\(f\)-derivation of \( X \). If \( df(x) = f(x) \land df(x) \) for all \( x \in X \), then for \( x = 0 \), \( df(0) = f(0) \land df(0) = 0 \land df(0) = df(0) \ast (df(0) \ast 0) = 0 \).

Conversely, if \( df(0) = 0 \), then \( df(x) = df(x \ast 0) = (f(x) \ast df(0)) \land (df(x) \ast f(0)) = (f(x) \ast 0) \land (df(x) \ast 0) = f(x) \land df(x) \), ending the proof. \( \square \)

**Proposition 2.11.** Let \( df \) be an \((l,r)\)-\(f\)-derivation of a BCI-algebra \( X \). Then,

(i) \( df(0) \in L_p(X) \), that is, \( df(0) = 0 \ast (0 \ast df(0)) \);

(ii) \( df(a) = df(0) \ast (0 \ast f(0)) = df(0) + f(a) \) for all \( a \in L_p(X) \);

(iii) \( df(a) \in L_p(X) \) for all \( a \in L_p(X) \);

(iv) \( df(a + b) = df(a) + df(b) - df(0) \) for all \( a, b \in L_p(X) \).
Proof. (i) The proof follows from Proposition 2.10(i).

(ii) Let \( a \in L_p(X) \), then \( a = 0 \ast (0 \ast a) \), and so \( f(a) = 0 \ast (0 \ast f(a)) \), that is, \( f(a) \in L_p(X) \). Hence

\[
d_f(a) = d_f(0 \ast (0 \ast a)) \\
= (d_f(0) \ast f(0 \ast a)) \land (f(0) \ast d_f(0 \ast a)) \\
= (d_f(0) \ast f(0 \ast a)) \land (0 \ast d_f(0 \ast a)) \\
= (0 \ast d_f(0 \ast a)) \ast ((0 \ast d_f(0 \ast a)) \ast (d_f(0) \ast f(0 \ast a))) \\
= (0 \ast d_f(0 \ast a)) \ast ((0 \ast (d_f(0) \ast f(0 \ast a))) \ast d_f(0 \ast a)) \\
= 0 \ast (0 \ast (d_f(0) \ast (0 \ast f(a)))) \\
= d_f(0) \ast (0 \ast f(a)) = d_f(0) + f(a).
\]

(iii) The proof follows directly from (ii).

(iv) Let \( a, b \in L_p(X) \). Note that \( a + b \in L_p(X) \), so from (ii), we note that

\[
d_f(a + b) = d_f(0) + f(a + b) \\
= d_f(0) + f(a) + d_f(0) + f(b) - d_f(0) = d_f(a) + d_f(b) - d_f(0).
\]

\[\square\]

**Proposition 2.12.** Let \( d_f \) be a \((r,l)\)-\( f \)-derivation of a BCI-algebra \( X \). Then,

(i) \( d_f(a) \in G(X) \) for all \( a \in G(X) \);

(ii) \( d_f(a) \in L_p(X) \) for all \( a \in G(X) \);

(iii) \( d_f(a) = f(a) \ast d_f(0) = f(a) + d_f(0) \) for all \( a \in L_p(X) \);

(iv) \( d_f(a + b) = d_f(a) + d_f(b) - d_f(0) \) for all \( a, b \in L_p(X) \).

Proof. (i) For any \( a \in G(X) \), we have

\[
d_f(a) = d_f(0 \ast (0 \ast a)) = (f(0) \ast d_f(0)) \land (d_f(0) \ast f(a)) \\
= (d_f(0) \ast f(0 \ast a)) \ast ((0 \ast d_f(0 \ast a)) \ast (0 \ast d_f(0 \ast a))) \\
= 0 \ast d_f(0 \ast a) \in L_p(X).
\]

(ii) For any \( a \in L_p(X) \), we get

\[
d_f(a) = d_f(a \ast 0) = (f(a) \ast d_f(0)) \land (d_f(a) \ast f(0)) \\
= d_f(a) \ast (d_f(a) \ast (f(a) \ast d_f(0))) = f(a) \ast d_f(0) \\
= f(a) \ast (0 \ast d_f(0)) = f(a) + d_f(0).
\]

(iv) The proof follows from (iii). This completes the proof. \[\square\]
Using Proposition 2.12, we know there is an \((l,r)-f\)-derivation which is not an \((r,l)-f\)-derivation as shown in the following example.

**Example 2.13.** Let \(\mathbb{Z}\) be the set of all integers and “−” the minus operation on \(\mathbb{Z}\). Then \((\mathbb{Z},−,0)\) is a BCI-algebra. Let \(d_f : X \to X\) be defined by \(d_f(x) = f(x) − 1\) for all \(x \in \mathbb{Z}\). Then,

\[
\begin{align*}
(d_f(x) − f(y)) \land (f(x) − d_f(y)) &= (f(x) − 1 − f(y)) \land (f(x) − (f(y) − 1)) \\
&= (f(x − y) − 1) \land (f(x − y) + 1) \\
&= (f(x − y) + 1) − 2 = f(x − y) − 1 \\
&= d_f(x − y).
\end{align*}
\]

Hence, \(d_f\) is an \((l,r)-f\)-derivation of \(X\). But \(d_f(0) = f(0) − 1 = −1 \neq 1 = f(0) − d_f(0) = 0 − d_f(0)\), that is, \(d_f(0) \notin G(X)\). Therefore, \(d_f\) is not an \((r,l)-f\)-derivation of \(X\) by Proposition 2.12(i).

### 3. Regular \(f\)-derivations

**Definition 3.1.** An \(f\)-derivation \(d_f\) of a BCI-algebra \(X\) is said to be regular if \(d_f(0) = 0\).

**Remark 3.2.** We know that the \(f\)-derivations \(d_f\) in Examples 2.5 and 2.7 are regular.

**Proposition 3.3.** Let \(X\) be a commutative BCI-algebra and let \(d_f\) be a regular \((r,l)-f\)-derivation of \(X\). Then the following hold.

(i) Both \(f(x)\) and \(d_f(x)\) belong to the same branch for all \(x \in X\).

(ii) \(d_f\) is an \((l,r)-f\)-derivation of \(X\).

**Proof.** (i) Let \(x \in X\). Then,

\[
0 = d_f(0) = d_f(x \ast x) \\
= (f(x) \ast f(x)) \land (d(x) \ast f(x)) \\
= (d(x) \ast f(x)) \ast ((d(x) \ast f(x)) \ast (f(x) \ast d_f(x))) \\
= (d(x) \ast f(x)) \ast ((d(x) \ast f(x)) \ast (f(x) \ast d_f(x))) \\
= f_x \ast d_f(x) \quad \text{since } f_x \ast d_f(x) \in L_p(X),
\]

and so \(f_x \leq d_f(x)\). This shows that \(d_f(x) \in V(f_x)\). Clearly, \(f(x) \in V(f_x)\).

(ii) By (i), we have \(f(x) \ast d_f(y) \in V(f_x \ast f_y)\) and \(d_f(x) \ast f(y) \in V(f_x \ast f_y)\). Thus \(d_f(x \ast y) = (f(x) \ast d_f(y)) \land (d_f(x) \ast f(y)) = (d_f(x) \ast f(y)) \land (f(x) \ast d_f(y))\), which implies that \(d_f\) is an \((l,r)-f\)-derivation of \(X\).

**Remark 3.4.** The \(f\)-derivations \(d_f\) in Examples 2.5 and 2.7 are regular \(f\)-derivations but we know that the \((l,r)-f\)-derivation \(d_f\) in Example 2.2 is not regular. In the following, we give some properties of regular \(f\)-derivations.

**Definition 3.5.** Let \(X\) be a BCI-algebra. Then define \(\ker d_f = \{x \in X \mid d_f(x) = 0\ \text{for all} \ f\text{-derivations } d_f\}\).
**Proposition 3.6.** Let $d_f$ be an $f$-derivation of a BCI-algebra $X$. Then the following hold:

(i) $d_f(x) \leq f(x)$ for all $x \in X$;
(ii) $d_f(x) * f(y) \leq f(x) * d_f(y)$ for all $x, y \in X$;
(iii) $d_f(x * y) = d_f(x) * f(y) \leq d_f(x) * d_f(y)$ for all $x, y \in X$;
(iv) ker $d_f$ is a subalgebra of $X$. Especially, if $f$ is monic, then ker $d_f \subseteq X_+.$

**Proof.** (i) The proof follows by Proposition 2.10(ii).

(ii) Since $d_f(x) \leq f(x)$ for all $x \in X$, then $d_f(x) * f(y) \leq f(x) * f(y) \leq f(x) * d_f(y)$.

(iii) For any $x, y \in X$, we have

$$d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$$

$$= (d_f(x) * f(y)) * ((d_f(x) * f(y)) * (f(x) * d_f(y)))$$

$$= (d_f(x) * f(y)) * 0 = d_f(x) * f(y) \leq d_f(x) * d_f(y),$$

which proves (iii).

(iv) Let $x, y \in ker d_f$, then $d_f(x) = 0 = d_f(y)$, and so $d_f(x * y) \leq d_f(x) * d_f(y) = 0 * 0 = 0$ by (iii), and thus $d_f(x * y) = 0$, that is, $x * y \in ker d_f$. Hence, ker $d_f$ is a subalgebra of $X$. Especially, if $f$ is monic, and letting $x \in ker d_f$, then $0 = d_f(x) \leq f(x)$ by (i), and so $f(x) \in X_+$, that is, $0 * f(x) = 0$, and thus $f(0 * x) = f(x)$, which implies that $0 * x = x$, and so $x \in X_+$, that is, ker $d_f \subseteq X_+.$

**Theorem 3.7.** Let $f$ be monic of a commutative BCI-algebra $X$. Then $X$ is $p$-semisimple if and only if ker $d_f = \{0\}$ for every regular $f$-derivation $d_f$ of $X$.

**Proof.** Assume that $X$ is $p$-semisimple BCI-algebra and let $d_f$ be a regular $f$-derivation of $X$. Then $X_+ = \{0\}$, and so ker $d_f = \{0\}$ by using Proposition 3.6(iv). Conversely, let ker $d_f = \{0\}$ for every regular $f$-derivation $d_f$ of $X$. Define a self-map $d_f^*$ of $X$ by $d_f^*(x) = f_x$ for all $x \in X$. Using Theorem 2.9, $d_f^*$ is an $f$-derivation of $X$. Clearly, $d_f^*(0) = f_0 = 0 * (0 * f(0)) = 0$, and so $d_f^*$ is a regular $f$-derivation of $X$. It follows from the hypothesis that ker $d_f^* = \{0\}$. In addition, $d_f^*(x) = f_x = 0 * (0 * f(x)) = f(0 * (0 * x)) = f(0) = 0$ for all $x \in X_+$, and thus $x \in ker d_f^*$, which shows that $X_+ \subseteq ker d_f^*$. Hence, by Proposition 3.6(iv), $X_+ = ker d_f^* = \{0\}$. Therefore $X$ is $p$-semisimple.

**Definition 3.8.** An ideal $A$ of a BCI-algebra $X$ is said to be an $f$-ideal if $f(A) \subseteq A$.

**Definition 3.9.** Let $d_f$ be a self-map of a BCI-algebra $X$. An $f$-ideal $A$ of $X$ is said to be $d_f$-invariant if $d_f(A) \subseteq A$.

**Theorem 3.10.** Let $d_f$ be a regular $(r,l)$-derivation of a BCI-algebra $X$, then every $f$-ideal $A$ of $X$ is $d_f$-invariant.

**Proof.** By Proposition 2.10(ii), we have $d_f(x) = f(x) \wedge d_f(x) \leq f(x)$ for all $x \in X$. Let $y \in d_f(A)$. Then $y = d_f(x)$ for some $x \in A$. It follows that $y * f(x) = d_f(x) * f(x) = 0 \in A$. Since $x \in A$, then $f(x) \in f(A) \subseteq A$ as $A$ is an $f$-ideal. It follows that $y \in A$ since $A$ is an ideal of $X$. Hence $d_f(A) \subseteq A$, and thus $A$ is $d_f$-invariant.

**Theorem 3.11.** Let $d_f$ be an $f$-derivation of a BCI-algebra $X$. Then $d_f$ is regular if and only if every $f$-ideal of $X$ is $d_f$-invariant.
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Proof. Let $d_f$ be a derivation of a BCI-algebra $X$ and assume that every $f$-ideal of $X$ is $d_f$-invariant. Then since the zero ideal $\{0\}$ is $f$-ideal and $d_f$-invariant, we have $d_f(\{0\}) \subseteq \{0\}$, which implies that $d_f(0) = 0$. Thus $d_f$ is regular. Combining this and Theorem 3.10, we complete the proof.

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