EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SEMILINEAR ELLIPTIC SYSTEM

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We consider the existence, the nonexistence, and the uniqueness of solutions of some special systems of nonlinear elliptic equations with boundary conditions. In a particular case, the system reduces to the homogeneous Dirichlet problem for the biharmonic equation $\Delta^2 u = |u|^p$ in a ball with $p > 0$.

1. Introduction

In this paper, we are interested in the existence, the nonexistence, and the uniqueness question for the following problem:

\begin{align*}
\Delta u &= |v|^{q-1}v \quad \text{in } B_R, \\
\Delta v &= |u|^p \quad \text{in } B_R, \\
u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_R,
\end{align*}

(1.1)

where $B_R$ denotes the open ball of radius $R$ centered at the origin in $\mathbb{R}^n$ ($n \geq 1$), $\partial/\partial \nu$ is the outward normal derivative, and $p, q > 0$.

Concerning uniqueness, we have the following theorem.

**Theorem 1.1.** (i) Let $p > 0$, $q \geq 1$ with $pq \neq 1$. Then (1.1) has at most one nontrivial radial solution $(u, v) \in (C^2(\overline{B}_R))^2$.

(ii) Let $p > 0$, $q \geq 1$ with $pq = 1$. Assume that (1.1) has a nontrivial radial solution $(u, v) \in (C^2(\overline{B}_R))^2$. Then all nontrivial radial solutions are given by $(\theta^q u, \theta v)$, where $\theta > 0$ is an arbitrary constant.

When $q = 1$ and $p \in (0, 1) \cup (1, \infty)$, Theorem 1.1 was established in [4] (see also the references therein). When $n = 1$, $q = 1$, and $p > 1$, the uniqueness of a nontrivial solution follows from a general result given in [5].

When $q = 1$, $p > 1$, and

\[ p < \frac{n+4}{n-4} \quad \text{if } n \geq 5, \]

(1.2)
the existence of a nontrivial solution was proved in [2, 5, 11]. The case \( q = 1 \) and \( 0 < p < 1 \) is well known: see, for instance, [4, 6]. Moreover, when \( q = 1 \), any nontrivial solution of (1.1) is positive in \( B_R \) because the Green function of \( \Delta^2 \) with Dirichlet boundary conditions is positive in \( B_R \) [1, 8]. Then it was proved in [2, 11, 12] that problem (1.1) has no nontrivial solutions, whether radial or not, if

\[
p \geq \frac{n+4}{n-4} \quad (n \geq 5).
\]

We will prove a nonexistence result and an existence result.

**Theorem 1.2.** Suppose \( n \geq 3 \). Let \( p, q > 0 \) satisfy

\[
\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}.
\]

(i) Let \( (u, v) \in (C^2(B_R))^2 \) be a solution of problem (1.1) such that \( u \geq 0 \) in \( B_R \). Then \( u = v = 0 \).

(ii) If \( (u, v) \in (C^2(B_R))^2 \) is a radial solution of problem (1.1), then \( u = v = 0 \).

**Theorem 1.3.** (i) Let \( p > 0, q \geq 1 \) with \( pq \neq 1 \) satisfy

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n} \quad \text{if} \quad n \geq 3.
\]

Then (1.1) has a nontrivial radial solution \( (u, v) \in (C^2(B_R))^2 \).

(ii) Let \( p > 0, q \geq 1 \) with \( pq = 1 \). Then there exists \( R > 0 \) such that (1.1) has a nontrivial radial solution \( (u, v) \in (C^2(B_R))^2 \).

**Remark 1.4.** Notice that when \( pq \leq 1 \), (1.5) holds.

In the sequel, \( \Delta \) denotes equally the Cartesian and the polar form of the Laplacian.

In Section 2, we give some preliminary results. Theorem 1.1 is proved in Section 3 using the same approach as in [4, 7]. In Section 4, we prove Theorem 1.2. We prove Theorem 1.3 in Section 5: the proof is based on a two-dimensional shooting argument for the ordinary differential equations associated to radial solutions of (1.1) [3, 5, 7, 15, 16]. The fact that \( q \geq 1 \) is crucial in the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

In this section, we first examine the structure of nontrivial radial solutions of (1.1).

**Lemma 2.1.** Let \( (u, v) \in (C^2(B_R))^2 \) be a nontrivial radial solution of (1.1). Then \( u' < 0 \) on \( (0, R) \), \( \Delta u(R) = u''(R) > 0 \) and \( v' > 0 \) on \( (0, R) \), \( v(0) < 0 < v(R) \).

**Proof.** Clearly \( u = 0 \) if and only if \( v = 0 \). We have

\[
r^{n-1} v'(r) = \int_0^r s^{n-1} |u'(s)|^p ds \geq 0, \quad 0 \leq r \leq R.
\]

(2.1)

Assume that \( v(0) \geq 0 \). Then (2.1) implies that \( v \geq 0 \) on \([0, R] \), hence \( \Delta u \geq 0 \) on \([0, R] \). Therefore \( r^{n-1} u'(r) \) is nondecreasing in \([0, R]\). Since \( u'(0) = u'(R) = u(R) = 0 \), we deduce
that \( u = 0 \) and we reach a contradiction. The case where \( v(R) \leq 0 \) can be handled in the same way. Therefore we have \( v(0) < 0 < v(R) \). We claim that \( u(0) \neq 0 \). Indeed assume that \( u(0) = 0 \). Using (2.1) and the first equation in (1.1), we deduce that there exists \( R' \in (0, R) \) such that \( r^{n-1}u'(r) \) is nonincreasing in \([0, R']\) and nondecreasing in \([R', R]\). Since \( u'(0) = u'(R) = 0 \), we obtain that \( u' \leq 0 \) in \([0, R]\). Using the fact that \( u'(0) = u'(R) = 0 \), we deduce that \( u < 0 \) in \((0, R)\). □

**Lemma 2.2.** Assume that \( n \geq 1 \) and \( p, q > 0 \). Let \( \alpha, \beta > 0 \) be fixed. If \((u, v) \in (C^2(\mathbb{R}^n))^2\) is a radial solution of

\[
\begin{align*}
\Delta u &= |v|^{q-1}v, \quad r > 0, \\
\Delta v &= |u|^p, \quad r > 0, \\
u(0) &= \alpha, \quad v(0) = -\beta, \quad u'(0) = v'(0) = 0
\end{align*}
\]  

(2.2)

such that \( uu' < 0 \) on \((0, \infty)\), then \( v < 0 \) on \((0, \infty)\).

**Proof.** We have \( 0 < u \leq \alpha \) on \([0, \infty)\). Therefore

\[
r^{n-1}u'(r) = \int_0^r s^{n-1}u(s) ds > 0 \quad \text{for } r > 0.
\]

(2.3)

Assume that the conclusion of the lemma is false. Then (2.3) implies that there exist \( a, b > 0 \) such that

\[
v(r) \geq a \quad \text{for } r \geq b.
\]

(2.4)

We deduce that

\[
(r^{n-1}u'(r))' \geq a^q r^{n-1} \quad \text{for } r \geq b,
\]

(2.5)

hence

\[
r^{n-1}u'(r) \geq a^q r^n - b^n + b^{n-1}u'(b) \quad \text{for } r \geq b,
\]

(2.6)

which implies that \( u'(r) > 0 \) for \( r \) large and we reach a contradiction. □

Now we give a lemma which is needed in the proof of Theorem 1.3.

**Lemma 2.3.** Assume that \( n \geq 1 \) and \( p, q > 0 \). Let \( \alpha, \beta > 0 \) be fixed. Assume that for some \( a > 0 \), \((u, v) \in (C^2(\overline{B}_a))^2\) is a radial solution of

\[
\begin{align*}
\Delta u &= |v|^{q-1}v \quad \text{in } [0, a], \\
\Delta v &= |u|^p \quad \text{in } [0, a], \\
u(0) &= \alpha, \quad v(0) = -\beta, \quad u'(0) = v'(0) = 0
\end{align*}
\]

(2.7)
such that \( uu' < 0 \) on \((0, a)\). Then
\[
|v(r)| \leq d \max \left( \beta, \alpha^{(p+1)/(q+1)} \right), \quad 0 \leq r \leq a,
\]
where
\[
d = \left( 1 + \frac{q+1}{p+1} \right)^{1/(q+1)}.
\]

Proof. We have \( 0 < u \leq \alpha \) on \([0, a)\). As in Lemma 2.2 we deduce that \( v' > 0 \) on \((0, a)\). We have
\[
\int_0^r (v' \Delta u + u' \Delta v) \, ds = \int_0^r (|v|^{q-1} v' + u^p u') \, ds
\]
for \( r \in [0, a] \). Since
\[
\int_0^r (v' \Delta u + u' \Delta v) \, ds = \int_0^r (u' v')' \, ds + 2(n-1) \int_0^r \frac{u'(s)v'(s)}{s} \, ds,
\]
we obtain
\[
\frac{|v(r)|^{q+1}}{q+1} + \frac{u(r)^{p+1}}{p+1} = \beta^{q+1} + \alpha^{p+1} + u'(r)v'(r) + 2(n-1) \int_0^r \frac{u'(s)v'(s)}{s} \, ds
\]
for \( r \in [0, a] \), which implies that
\[
|v(r)|^{q+1} \leq \beta^{q+1} + \frac{q+1}{p+1} \alpha^{p+1}, \quad 0 \leq r \leq a,
\]
and the lemma follows.

3. Proof of Theorem 1.1

(i) Let \((u, v)\) and \((w, z)\) be two nontrivial radial solutions of (1.1). Let \( s \) and \( t \) be defined by
\[
s = 2 \frac{q+1}{pq-1}, \quad t = 2 \frac{p+1}{pq-1}.
\]
For \( \lambda > 0 \) we set
\[
\tilde{w}(r) = \lambda^s w(\lambda r), \quad \tilde{z}(r) = \lambda^t z(\lambda r), \quad 0 \leq r \leq \frac{R}{\lambda}.
\]
By Lemma 2.1, $\tilde{w} > 0$ on $[0, R/\lambda)$ and then we have
\[
\Delta \tilde{w}(r) = |\tilde{z}(r)|^{q-1}z(r), \quad 0 \leq r \leq \frac{R}{\lambda},
\]
\[
\Delta \tilde{z}(r) = \tilde{w}(r)^p, \quad 0 \leq r \leq \frac{R}{\lambda},
\]
\[
\tilde{w}\left(\frac{R}{\lambda}\right) = \tilde{w}'\left(\frac{R}{\lambda}\right) = 0.
\]
(3.3)

Choose $\lambda$ such that $\lambda \tilde{w}(0) = u(0)$. Then we have
\[
\tilde{w}(0) = u(0).
\]
(3.4)

We want to show that
\[
\tilde{z}(0) = v(0).
\]
(3.5)

Suppose that $\tilde{z}(0) < v(0)$. If there exists $a \in (0, \min(R, R/\lambda)]$ such that $\tilde{z} - v < 0$ on $[0, a)$ and $(\tilde{z} - v)(a) = 0$, then $\Delta(\tilde{z} - v) < 0$ on $[0, a)$. Equation (3.4) and the maximum principle imply that $\tilde{w} - u < 0$ on $(0, a]$. Therefore $\Delta(\tilde{z} - v) < 0$ on $(0, a]$ and the maximum principle implies that $\tilde{z} - v > (\tilde{z} - v)(a) = 0$ on $[0, a)$, a contradiction. Thus $\tilde{z} - v < 0$ on $[0, \min(R, R/\lambda)]$. Then, as before, we show that $\tilde{w} - u < 0$ on $(0, \min(R, R/\lambda)]$. Since
\[
(\tilde{w} - u)\left(\min(\frac{R}{\lambda}, R)\right) = \begin{cases} 
-u\left(\frac{R}{\lambda}\right) & \text{if } \lambda > 1, \\
0 & \text{if } \lambda = 1, \\
\tilde{w}(R) & \text{if } \lambda < 1,
\end{cases}
\]
(3.6)

we deduce that $\lambda > 1$ with the help of Lemma 2.1. Now using the fact that $r^{q-1}(\tilde{w} - u)'(r)$ is decreasing in $[0, R/\lambda]$, we get $(\tilde{w} - u)'(R/\lambda) < 0$. Since $(\tilde{w} - u)'(R/\lambda) = -u'(R/\lambda) > 0$ by Lemma 2.1, we again obtain a contradiction. The case $\tilde{z}(0) > v(0)$ can be handled in the same way. Thus (3.5) is proved.

Now we define the functions $U$, $W$, $F$, and $G_n$ by
\[
U(r) = (u(r), v(r)), \quad 0 \leq r \leq R,
\]
\[
W(r) = (\tilde{w}(r), \tilde{z}(r)), \quad 0 \leq r \leq \frac{R}{\lambda},
\]
\[
F(x, y) = (|y|^{q-1}y, x^p), \quad x \geq 0, \ y \in \mathbb{R},
\]
\[
G_n(r, s) = \begin{cases} 
\frac{r - s}{s} & \text{if } n = 1, \\
\ln\left(\frac{r}{s}\right) & \text{if } n = 2, \\
\frac{s}{n-2} \left(1 - \left(\frac{s}{r}\right)^{n-2}\right) & \text{if } n \geq 3
\end{cases}
\]
(3.7)
for \(0 \leq s \leq r\). Using (3.4), (3.5), and the fact that \(u'(0) = \tilde{w}'(0) = v'(0) = \tilde{z}'(0) = 0\), we easily obtain

\[
U(r) - W(r) = \int_0^r G_n(r,s)(F(U(s)) - F(W(s)))ds
\]

for \(r \in [0, \min(R,R/\lambda)]\). When \(p \geq 1\), \(F\) is locally Lipschitz continuous, and using Gronwall’s lemma we obtain \(U = W\) on \([0, \min(R,R/\lambda)]\). When \(p < 1\), let \(a \in (0, \min(R, R/\lambda))\) be fixed. Then \(u(0) \geq u(r) \geq u(a) > 0\), \(\tilde{w}(0) = u(0) \geq \tilde{w}(r) \geq \tilde{w}(a) > 0\) for \(r \in [0,a]\). Since \(F\) is locally Lipschitz continuous on \((0, +\infty) \times \mathbb{R}\), as before we obtain \(U = W\) on \([0,a]\). By continuity we get \(U = W\) on \([0, \min(R,R/\lambda)]\). Now we deduce that \(\lambda = 1\) and thus \((u,v) = (w,z)\) on \([0,R]\).

(ii) Let \((u,v)\) be a nontrivial radial solution of problem (1.1). Then, for any \(\theta > 0\), \((w,z) = (\theta^q u, \theta v)\) is a nontrivial radial solution of problem (1.1). Now let \((w,z)\) be a nontrivial radial solution of (1.1). Choose \(\theta > 0\) such that \(\theta^q u(0) = w(0)\) and define \(\tilde{w} = \theta^q u\), \(\tilde{z} = \theta v\). Then \((\tilde{w}, \tilde{z})\) is a nontrivial radial solution of (1.1) such that \(\tilde{w}(0) = w(0)\). Arguing as in part (i), we show that \(\tilde{z}(0) = z(0)\) and that \((\tilde{w}, \tilde{z}) = (w,z)\).

**Remark 3.1.** Our technique also applies when there is a homogeneous dependence on the radius \(|x|\). More precisely, for \(p > 0\), \(q \geq 1\), and \(pq \neq 1\), the following system

\[
\begin{align*}
\Delta u &= |x|^\mu |v|^{q-1}v \quad \text{in } B_R, \\
\Delta v &= |x|^\nu |u|^p \quad \text{in } B_R, \\
u &= \frac{\partial u}{\partial v} = 0 \quad \text{on } \partial B_R,
\end{align*}
\]

where \(\mu, \nu \geq 0\), has at most one nontrivial radial solution \((u,v)\). Indeed, the arguments are the same with \(s\) and \(t\) in (2.1) replaced by

\[
s = \frac{2(q+1) + \nu + q\mu}{pq - 1}, \quad t = \frac{2(p+1) + \mu + p\nu}{pq - 1}.
\]

Now let \(p > 0\), \(q \geq 1\) with \(pq = 1\). Assume that problem (3.10) has a nontrivial radial solution \((u,v)\). Then all nontrivial radial solutions are given by \((\theta^q u, \theta v)\), where \(\theta > 0\) is an arbitrary constant.

**4. Proof of Theorem 1.2**

(i) Let \((u,v) \in (C^2(\overline{B}_R))^2\) be a solution of problem (1.1) such that \(u \geq 0\) in \(B_R\). We have \(x \cdot v(x) = R\) for all \(x \in \partial B_R\). Multiplying the first equation in (1.1) by \(x \cdot \nabla v\) and integrating over \(B_R\), we get

\[
\int_{B_R} (x \cdot \nabla v) \Delta u \, dx = \int_{B_R} (x \cdot \nabla v) |v|^{q-1} v \, dx.
\]

Integrating by parts, we obtain

\[
\int_{B_R} (x \cdot \nabla v) |v|^{q-1} v \, dx = -\frac{n}{q+1} \int_{B_R} |v|^{q+1} \, dx + \frac{R}{q+1} \int_{\partial B_R} |v|^{q+1} \, d\sigma.
\]
Similarly we get
\[
\int_{B_R} (x \cdot \nabla u) \Delta v \, dx = \int_{B_R} (x \cdot \nabla u) u^p \, dx = -\frac{n}{p+1} \int_{B_R} u^{p+1} \, dx. \tag{4.3}
\]
Now we have
\[
\int_{B_R} ((x \cdot \nabla v) \Delta u + (x \cdot \nabla u) \Delta v) \, dx = (n-2) \int_{B_R} \nabla u \cdot \nabla v \, dx. \tag{4.4}
\]
Then we deduce that
\[
\frac{R}{q+1} \int_{\partial B_R} |v|^{q+1} \, d\sigma = \frac{n}{q+1} \int_{B_R} |v|^{q+1} \, dx + \frac{n}{p+1} \int_{B_R} u^{p+1} \, dx + (n-2) \int_{B_R} \nabla u \cdot \nabla v \, dx. \tag{4.5}
\]
Since
\[
\int_{B_R} \nabla u \cdot \nabla v \, dx = -\int_{B_R} v \Delta u \, dx = -\int_{B_R} |v|^{q+1} \, dx,
\]
\[
\int_{B_R} \nabla u \cdot \nabla v \, dx = -\int_{B_R} u \Delta v \, dx = -\int_{B_R} u^{p+1} \, dx,
\]
we can write
\[
\frac{R}{q+1} \int_{\partial B_R} |v|^{q+1} \, d\sigma = n \left( \frac{1}{p+1} + \frac{1}{q+1} - \frac{n-2}{n} \right) \int_{B_R} |v|^{q+1} \, dx. \tag{4.7}
\]
Using (1.4) we deduce that \( v = 0 \) on \( \partial B_R \). The maximum principle implies that \( v \leq 0 \) in \( B_R \). Therefore \( \Delta u \leq 0 \) in \( B_R \). The Hopf boundary point lemma implies that \( u = 0 \) in \( B_R \) and (i) is proved.

(ii) follows from (i) and Lemma 2.1.

Remark 4.1. Clearly Theorem 1.2(i) can be extended to more general domains and more general nonlinearities as in [2, 11, 12] and Theorem 1.2(ii) can be extended to more general nonlinearities.

5. Proof of Theorem 1.3

We will use a two-dimensional shooting argument for the ordinary differential equations associated to radial solutions of (1.1) [3, 5, 7, 15, 16]. We consider the one-dimensional (singular if \( n \geq 2 \)) initial value problem (2.2) where \( \alpha > 0, \beta > 0 \).

We will need a series of lemmas. We begin with a standard local existence and uniqueness result.

Lemma 5.1. For any \( \alpha > 0, \beta > 0 \) there exists \( T = T(\alpha, \beta) > 0 \) such that problem (2.2) on \([0, T]\) has a unique solution \((u, v) \in (C^2[0, T])^2\).
Proof. Let $\alpha, \beta > 0$ be given. Choose $T = T(\alpha, \beta) > 0$ such that
\[
T = \min \left( \left( \frac{n\alpha}{\beta^q} \right)^{1/2}, \left( \frac{n\beta}{\alpha^p} \right)^{1/2} \right),
\] (5.1)
and consider the set of functions
\[
Z = \left\{ (u, v) \in (C[0, T])^2; \frac{\alpha}{2} \leq u(r) \leq \alpha, \ -\beta \leq v(r) \leq -\frac{\beta}{2} \text{ for } 0 \leq r \leq T \right\}. \] (5.2)
Clearly $Z$ is a bounded closed convex subset of the Banach space $(C[0, T])^2$ endowed with the norm $\| (u, v) \| = \max(\|u\|_\infty, \|v\|_\infty)$. Define
\[
L(u, v)(r) = \left( \alpha + \int_0^r G_n(r, s) |v(s)|^{q-1}v(s) \, ds, -\beta + \int_0^r G_n(r, s) |u(s)|^{p} \, ds \right)
\] (5.3)
for $r \in [0, T]$ and $(u, v) \in (C[0, T])^2$, where $G_n$ is defined in (3.8). It is easily verified that $L$ is a compact operator mapping $Z$ into itself, and so there exists $(u, v) \in Z$ such that $(u, v) = L(u, v)$ by the Schauder fixed point theorem. Clearly $(u, v) \in (C^2[0, T])^2$ and $(u, v)$ is a solution of (2.2) on $[0, T]$. Since the right-hand side in (2.2) is Lipschitz continuous in $(u, v) \in [\alpha^2, \alpha] \times [-\beta, -\beta/2]$, the uniqueness follows. \hfill \□

Remark 5.2. Notice that $u(r) > 0$ and $v(r) < 0$ for $r \in [0, T]$. Then direct integration of the system (2.2) implies that $u' < 0$ and $v' > 0$ in $(0, T]$.

In view of Lemma 5.1, for any $\alpha, \beta > 0$ problem (2.2) has a unique local solution: let $[0, R_{\alpha, \beta})$ denote the maximum interval of existence of that solution ($R_{\alpha, \beta} = +\infty$ possibly). If $0 < p < 1$, the uniqueness of the solution could fail at any point $r$ where $u(r) = 0$. In this case, $R_{\alpha, \beta}$ could also depend on the particular solution itself. Define
\[
P_{\alpha, \beta} = \{ s \in (0, R_{\alpha, \beta}); u(\alpha, \beta, r) u'(\alpha, \beta, r) < 0 \ \forall r \in (0, s) \}, \] (5.4)
where $(u(\alpha, \beta, \cdot), v(\alpha, \beta, \cdot))$ is a solution of (2.2) in $[0, R_{\alpha, \beta})$. $P_{\alpha, \beta} \neq \emptyset$ by Remark 5.2. Set
\[
r_{\alpha, \beta} = \sup P_{\alpha, \beta}. \] (5.5)
Notice that the solution is unique on $[0, r_{\alpha, \beta}]$, so $r_{\alpha, \beta}$ depends only on $\alpha, \beta$.

Lemma 5.3. $u'(\alpha, \beta, r) < 0$ for $r \in (0, r_{\alpha, \beta})$ and $v'(\alpha, \beta, r) > 0$ for $r \in (0, R_{\alpha, \beta})$.

Proof. The first assertion follows from the definition of $r_{\alpha, \beta}$. Since $u(\alpha, \beta, r) > 0$ for $r \in [0, r_{\alpha, \beta})$, integrating the second equation in (2.2) from 0 to $r \in (0, R_{\alpha, \beta})$ we obtain $v'(\alpha, \beta, r) > 0$ for $r \in (0, R_{\alpha, \beta})$. \hfill \□

Lemma 5.4. For any $\alpha, \beta > 0, r_{\alpha, \beta} < \infty$.

Proof. Assume that $r_{\alpha, \beta} = \infty$. We easily get a contradiction when $n = 1$ or 2. Now if $n \geq 3$, we set $z = -v$. By Lemma 2.2, $z > 0$ on $[0, \infty)$ and we have
\[
\begin{align*}
-\Delta u &= z^2, & r > 0, \\
-\Delta z &= u^p, & r > 0.
\end{align*}
\] (5.6)
Since \( p, q \) satisfy (1.5), we obtain a contradiction with the help of the nonexistence results established in [9, 10, 13, 14].

**Lemma 5.5.** For any \( a \in [T(\alpha, \beta), r, \alpha] \), there exists \( b = b(\alpha, \beta, a) > 0 \) such that the maximal extension of \((u, v)\) includes the interval \([0, a + b]\). Moreover,

\[
b(\alpha, \beta, a) = \frac{m(\alpha, \beta)}{a + \sqrt{a^2 + m(\alpha, \beta)}},
\]

where

\[
m(\alpha, \beta) = \min \left( \frac{n\beta}{2^{p-1} \alpha^p}, \frac{na}{2^{q-1} d^q \left( \max(\beta, \alpha^{(p+1)/(q+1)}) \right)^q} \right),
\]

with \( d \) given in Lemma 2.3.

**Proof.** Lemma 5.5 is essentially a local existence result, with initial data \( u(a), v(a), u'(a), v'(a) \) at \( r = a \). Let

\[
W = \left\{ (u, v) \in (C[a, a + b])^2 ; \, |u(r) - u(a)| \leq \alpha, \, 0 \leq v(r) - v(a) \leq \beta \text{ for } a \leq r \leq a + b \right\},
\]

where \( b = b(\alpha, \beta, a) \) is given in the lemma. \( W \) is a bounded closed convex subset of the Banach space \((C[a, a + b])^2\) equipped with the norm \( \| (u, v) \| = \max(\| u \|_\infty, \| v \|_\infty) \). Consider the mapping \( S(u, v) = (S_1(u, v), S_2(u, v)) \) on \((C[a, a + b])^2\) given by

\[
S_1(u, v)(r) = u(a) + \int_a^r \frac{dt}{t^{n-1}} \int_0^t s^{n-1} |v(s)|^{q-1} v(s) ds,
\]

\[
S_2(u, v)(r) = v(a) + \int_a^r \frac{dt}{t^{n-1}} \int_0^t s^{n-1} |u(s)|^p ds
\]

for \( a \leq r \leq a + b \), where we also denote by \( u, v \) the unique solution of (2.2) on \([0, a]\). Let \((u, v) \in W\). Using Lemma 5.3, we have

\[
|u(s)| \leq u(a) + \alpha \leq 2\alpha, \quad s \in [a, a + b].
\]

Therefore we get

\[
0 \leq S_2(u, v)(r) - v(a) \leq 2^{p-1} \alpha^p \frac{r^2 - a^2}{n} \leq \beta, \quad r \in [a, a + b].
\]

By Lemma 2.3 we have

\[
|v(s)| \leq |v(a)| + \beta \leq 2d \max \left( \beta, \alpha^{(p+1)/(q+1)} \right), \quad s \in [a, a + b].
\]

Therefore for \( a \leq r \leq a + b \), we obtain

\[
|S_1(u, v)(r) - u(a)| \leq 2^{q-1} d^q \left( \max \left( \beta, \alpha^{(p+1)/(q+1)} \right) \right)^q \frac{r^2 - a^2}{n} \leq \alpha.
\]
We have thus proved that $S(W) \subset W$. Since $S$ is a compact operator, there exists $(u, v) \in W$ such that $(u, v) = S(u, v)$ by the Schauder fixed point theorem. Clearly $(u, v) \in (C^2[a, a + b])^2$ and $(u, v)$ is a solution of (2.2) on $[a, a + b]$ which extends the solution $(u, v)$ on $[0, a]$. □

**Lemma 5.6.** For any $\alpha, \beta > 0$,

$$R_{\alpha, \beta} \geq r_{\alpha, \beta} + \frac{m(\alpha, \beta)}{r_{\alpha, \beta} + \sqrt{r_{\alpha, \beta}^2 + m(\alpha, \beta)}}. \tag{5.15}$$

**Proof.** By Lemma 5.5, for any $a \in (T(\alpha, \beta), r_{\alpha, \beta})$ we have

$$R_{\alpha, \beta} > a + \frac{m(\alpha, \beta)}{a + \sqrt{a^2 + m(\alpha, \beta)}}. \tag{5.16}$$

The lemma follows by letting $a \to r_{\alpha, \beta}$. □

**Proposition 5.7.** For any $\alpha > 0$, there exists a unique $\beta > 0$ such that $u(\alpha, \beta, r_{\alpha, \beta}) = u'(\alpha, \beta, r_{\alpha, \beta}) = 0$. \(□\)

**Proof.** We first prove the uniqueness. Let $\alpha > 0$ be fixed. Suppose that there exist $\beta > y > 0$ such that $u(\alpha, \beta, r_{\alpha, \beta}) = u'(\alpha, \beta, r_{\alpha, \beta}) = u(\alpha, y, r_{\alpha, y}) = u'(\alpha, y, r_{\alpha, y}) = 0$. Using the same arguments as in the proof of (3.5) we obtain a contradiction.

Now we prove the existence. Suppose that there exists $\alpha > 0$ such that for any $\beta > 0$ $u(\alpha, \beta, r_{\alpha, \beta}) > 0$ or $u'(\alpha, \beta, r_{\alpha, \beta}) < 0$. Define the sets

- \(B = \{ \beta > 0; u(\alpha, \beta, r_{\alpha, \beta}) = 0, u'(\alpha, \beta, r_{\alpha, \beta}) < 0 \}\),
- \(C = \{ \beta > 0; u(\alpha, \beta, r_{\alpha, \beta}) > 0, u'(\alpha, \beta, r_{\alpha, \beta}) = 0 \}\). \(\tag{5.17}\)

The proof of the proposition is completed by using the next two lemmas which contradict the fact that

\[(0, +\infty) = B \cup C. \tag{5.18}\]

**Lemma 5.8.** (i) Suppose $B \neq \emptyset$. Then there exists $m > 0$ such that $m \leq \inf B$.

(ii) Suppose $C \neq \emptyset$. Then there exists $M > 0$ such that $M \geq \sup C$.

**Lemma 5.9.** $B$ and $C$ are open.

**Proof of Lemma 5.8.** We have

$$u(\alpha, \beta, r) = \alpha + \int_0^r G_n(r, s) \left| v(\alpha, \beta, s) \right|^{q-1} v(\alpha, \beta, s) ds, \quad 0 \leq r < R_{\alpha, \beta}, \tag{5.19}$$

$$v(\alpha, \beta, r) = -\beta + \int_0^r G_n(r, s) \left| u(\alpha, \beta, s) \right|^p ds, \quad 0 \leq r < R_{\alpha, \beta}. \tag{5.20}$$
(i) Let $\beta \in B$. Assume first that $v(\alpha, \beta, \cdot) < 0$ on $[0, \alpha, \beta)$. Then Lemma 5.3 and (5.19) imply

$$r_{\alpha, \beta} \geq \left(\frac{2n\alpha}{\beta^q}\right)^{1/2}.$$  \hspace{1cm} (5.21)

Now, if there exists $s_{\alpha, \beta} \in [0, r_{\alpha, \beta})$ such that $v(\alpha, \beta, s) = 0$, Lemma 5.3 implies that $-\beta \leq v(\alpha, \beta, \cdot) < 0$ in $[0, s_{\alpha, \beta})$ and $v(\alpha, \beta, \cdot) > 0$ in $(s_{\alpha, \beta}, r_{\alpha, \beta}]$. Then from (5.19) we get

$$\alpha = -\int_0^{r_{\alpha, \beta}} G_n(r_{\alpha, \beta}, s) \left| v(\alpha, \beta, s) \right|^{q-1} v(\alpha, \beta, s) ds \leq \int_0^{s_{\alpha, \beta}} G_n(r_{\alpha, \beta}, s) ds \leq \beta^q \int_0^{s_{\alpha, \beta}} G_n(r_{\alpha, \beta}, s) ds \leq \beta^q \frac{r_{\alpha, \beta}^2}{2n},$$

and (5.21) still holds.

Suppose that $\inf B = 0$ and let $(\beta_j)$ be a sequence in $B$ decreasing to zero. Then $r_{\alpha, \beta_j} \to +\infty$ by (5.21). Let $r > 0$ be fixed. We can assume that $r_{\alpha, \beta_j} > r$ for all $j$. If $v(\alpha, \beta_j, s) < 0$ for $s \in [0, r]$, we have

$$u(\alpha, \beta_j, r) = \alpha - \int_0^r G_n(r, s) \left| v(\alpha, \beta_j, s) \right|^q ds \geq \alpha - \frac{r^2 \beta_j^q}{2n}.$$ \hspace{1cm} (5.23)

If $s_{\alpha, \beta_j} < r$, we have

$$u(\alpha, \beta_j, r) = \alpha - \int_0^{s_{\alpha, \beta_j}} G_n(r, s) \left| v(\alpha, \beta_j, s) \right|^q ds + \int_{s_{\alpha, \beta_j}}^r G_n(r, s) v(\alpha, \beta_j, s)^q ds \geq \alpha - \frac{r^2 \beta_j^q}{2n}.$$ \hspace{1cm} (5.24)

Therefore using Lemma 5.3 we obtain

$$u(\alpha, \beta_j, s) \geq \alpha - \frac{r^2 \beta_j^q}{2n} \quad \text{for} \ s \in [0, r],$$ \hspace{1cm} (5.25)

from which we deduce that

$$u(\alpha, \beta_j, s) \geq \frac{\alpha}{2}$$ \hspace{1cm} (5.26)

for $s \in [0, r]$ and $j$ large. From (5.20) we get

$$v(\alpha, \beta_j, r) \geq -\beta_j + \frac{r^2 \alpha^p}{2p+1 \cdot n}$$ \hspace{1cm} (5.27)

for $j$ large. Thus if we choose $r$ such that

$$-\beta_j + \frac{r^2 \alpha^p}{2p+1 \cdot n} \geq 1,$$ \hspace{1cm} (5.28)
using Lemma 5.3 we get
\[ v(\alpha, \beta_j, s) \geq 1 \] (5.29)
for \( r \leq s \leq r_{a\beta_j} \) and \( j \) large. We also have
\[ -\beta_j \leq v(\alpha, \beta_j, s) \leq -\beta_j + \frac{r^2 \alpha^p}{2n} \] (5.30)
for \( s \in [0, r] \). Therefore there exists \( c > 0 \) such that
\[ |v(\alpha, \beta_j, s)| \leq c \] (5.31)
for \( s \in [0, r] \) and all \( j \). There exists \( k > 0 \) such that
\[ \int_r^{r_{a\beta_j}} G_n(r_{a\beta_j}, s) \, ds \geq kr_{a\beta_j}^2 \] (5.32)
for \( j \) large. Now we write
\[
\alpha = -\int_0^{r_{a\beta_j}} G_n(r_{a\beta_j}, s) \, |v(\alpha, \beta_j, s)|^{q-1} v(\alpha, \beta_j, s) \, ds
\]
\[
= -\int_0^r G_n(r_{a\beta_j}, s) \, |v(\alpha, \beta_j, s)|^{q-1} v(\alpha, \beta_j, s) \, ds
\]
\[
- \int_r^{r_{a\beta_j}} G_n(r_{a\beta_j}, s) v(\alpha, \beta_j, s)^q \, ds
\]
\[
\leq c^q \int_0^r G_n(r_{a\beta_j}, s) \, ds - \int_r^{r_{a\beta_j}} G_n(r_{a\beta_j}, s) \, ds
\]
\[
\leq c^q rr_{a\beta_j} - kr_{a\beta_j}^2
\]
for \( j \) large, where we have used the fact that \( G_n(r_{a\beta_j}, s) \leq r_{a\beta_j} - s \) for \( 0 \leq s \leq r_{a\beta_j} \). Since the last term above tends to \(-\infty\), we get a contradiction.

(ii) Let \( \beta \in C \). We claim that \( v(\alpha, \beta, r_{a\beta}) > 0 \). If not, by Lemma 5.3 we have \( \Delta u(\alpha, \beta, \cdot) < 0 \) on \([0, r_{a\beta}]\) for some \( \beta \in C \). Since \( u'(\alpha, \beta, 0) = 0 \), we obtain \( u'(\alpha, \beta, r_{a\beta}) < 0 \), a contradiction. Therefore (5.20) implies
\[ \beta < \int_0^{r_{a\beta}} G_n(r_{a\beta}, s) u(\alpha, \beta, s)^p \, ds \] (5.34)
for \( \beta \in C \). Suppose that \( \sup C = +\infty \) and let \( \beta_j \) be a sequence in \( C \) increasing to \( +\infty \). Since \( 0 < u(\alpha, \beta_j, r) \leq \alpha \) for \( r \in [0, r_{a\beta_j}] \), (5.34) implies that \( r_{a\beta_j} \to +\infty \) as \( j \to +\infty \). Then we can assume that \( r_{a\beta_j} \geq 1 \) and that \( \alpha^p \leq \beta_j \) for all \( j \). From (5.20) we get
\[ -\beta_j \leq v(\alpha, \beta_j, r) \leq -\frac{2n-1}{2n} \beta_j \leq -\frac{\beta_j}{2} \quad \text{for } r \in [0, 1], \] (5.35)
and using (5.19) we deduce that \( u(\alpha, \beta_j, 1) \leq \alpha - \beta_j^q/n2^{q+1} \). But then \( u(\alpha, \beta_j, 1) < 0 \) for \( j \) large and we reach a contradiction. \( \square \)
Remark 5.10. The proof above shows that, when $\beta \in C$, there exists $s_{a, \beta} \in (0, r_{a, \beta})$ such that $v(\alpha, \beta, \cdot) < 0$ on $[0, s_{a, \beta})$ and $v(\alpha, \beta, \cdot) > 0$ on $(s_{a, \beta}, r_{a, \beta}]$. When $\beta \in B$, $s_{a, \beta}$ may not exist.

Proof of Lemma 5.9

Case 1 ($p \geq 1$). Then the right-hand side of (2.2) is Lipschitz continuous. Let $\beta \in B$. We have $u(\alpha, \beta, r_{a, \beta}) = 0$ and $u'(\alpha, \beta, r_{a, \beta}) < 0$. Therefore we can find $\epsilon > 0$ such that

$$u(\alpha, \beta, r_{a, \beta} + \epsilon) < 0, \quad u'(\alpha, \beta, r_{a, \beta} + \epsilon) < 0. \quad (5.36)$$

But then by continuous dependence on initial data, there exists $\eta > 0$ such that

$$u(\alpha, \gamma, r_{a, \beta} + \epsilon) < 0, \quad u'(\alpha, \gamma, r_{a, \beta} + \epsilon) < 0 \quad (5.37)$$

for $|\gamma - \beta| < \eta$. The first inequality in (5.37) implies that there exists $x \in (0, r_{a, \beta} + \epsilon)$ such that $u(\alpha, \gamma, x) = 0$ and $u(\alpha, \gamma, r) > 0$ for $r \in [0, x)$. $\Delta v(\alpha, \gamma, r) > 0$ for $r \in [0, x)$ and $\Delta v(\alpha, \gamma, r) \geq 0$ for $r \in [x, r_{a, \beta} + \epsilon]$. Then $v'(\alpha, \gamma, r) > 0$ for $r \in (0, r_{a, \beta} + \epsilon)$ and $v(\alpha, \gamma, \cdot)$ is increasing on $[0, r_{a, \beta} + \epsilon]$. We deduce that $\Delta u(\alpha, \gamma, \cdot)$ is increasing on $[0, r_{a, \beta} + \epsilon]$. If $\Delta u(\alpha, \gamma, r_{a, \beta} + \epsilon) \leq 0$, then $u'(\alpha, \gamma, r) < 0$ for $r \in (0, r_{a, \beta} + \epsilon)$. If $\Delta u(\alpha, \gamma, r_{a, \beta} + \epsilon) > 0$, then there exists $s_{a, \gamma} \in (0, r_{a, \beta} + \epsilon)$ such that $\Delta u(\alpha, \gamma, \cdot) < 0$ in $[0, s_{a, \gamma})$ and $\Delta u(\alpha, \gamma, \cdot) > 0$ in $(s_{a, \gamma}, r_{a, \beta} + \epsilon)$. We deduce that $u'(\alpha, \gamma, \cdot)$ is decreasing (resp., increasing) in $[0, s_{a, \gamma}]$ (resp., $(s_{a, \gamma}, r_{a, \beta} + \epsilon)$). Since $u'(\alpha, \gamma, 0) = 0$, the second inequality in (5.37) implies that $u'(\alpha, \gamma, r) < 0$ for $r \in (0, r_{a, \beta} + \epsilon)$. Therefore $x = r_{a, \gamma}$ for $|\gamma - \beta| < \eta$ and $(\beta - \eta, \beta + \eta) \subset B$. Thus $B$ is open. Now let $\beta \in C$. We have $u(\alpha, \beta, r_{a, \beta}) > 0$ and $u'(\alpha, \beta, r_{a, \beta}) = 0$. By Remark 5.10, we have $v(\alpha, \beta, r_{a, \beta}) > 0$, hence $\Delta u(\alpha, \beta, r_{a, \beta}) = u''(\alpha, \beta, r_{a, \beta}) > 0$. Therefore we can find $\epsilon > 0$ such that

$$u(\alpha, \beta, r) > 0, \quad r \in [0, r_{a, \beta} + \epsilon], \quad u'(\alpha, \beta, r_{a, \beta} + \epsilon) > 0. \quad (5.38)$$

Then by continuous dependence on initial data, there exists $\eta > 0$ such that

$$u(\alpha, \gamma, r) > 0, \quad r \in [0, r_{a, \beta} + \epsilon], \quad u'(\alpha, \gamma, r_{a, \beta} + \epsilon) > 0 \quad (5.39)$$

for $|\gamma - \beta| < \eta$. The second inequality in (5.39) implies that there exists $x \in (0, r_{a, \beta} + \epsilon)$ such that $u'(\alpha, \gamma, x) = 0$ and $u'(\alpha, \gamma, r) < 0$ for $r \in (0, x)$. Therefore $x = r_{a, \gamma}$ for $|\gamma - \beta| < \eta$ and $(\beta - \eta, \beta + \eta) \subset C$. Thus $C$ is open.

Case 2 ($0 < p < 1$). We first show that $C$ is open. Indeed let $\beta \in C$. Since $u(\alpha, \beta, r) > 0$ for $r \in (0, r_{a, \beta}]$, the system (2.2) is Lipschitz continuous in $u$ and $v$ when $u$ is in a neighborhood of the interval $[u(\alpha, \beta, r_{a, \beta}), \alpha]$ in $(0, \infty)$, and the solution $u(\alpha, \beta, \cdot)$, $v(\alpha, \beta, \cdot)$ can be uniquely extended to $[0, r_{a, \beta} + t]$ for some $t > 0$, with $u(\alpha, \beta, r) > 0$ for $r \in [0, r_{a, \beta} + t]$. Then we can argue as in Case 1. Now we show that $B$ is open. As in [15], this case is much more difficult. We begin with the following two steps. Let $\beta \in B$.

Step 1. There exists $c > 0$ and $\eta > 0$ such that when $|\beta - \gamma| < \eta$, the solutions $u(\alpha, \gamma, \cdot)$, $v(\alpha, \gamma, \cdot)$, and $u(\alpha, \beta, \cdot)$, $v(\alpha, \beta, \cdot)$ are defined on $[0, r_{a, \beta} + c]$. 

By Lemma 5.6, \( u(\alpha, \beta, \cdot), v(\alpha, \beta, \cdot) \) can be extended to the interval \([0, r_{a,\beta} + b(\alpha, \beta, r_{a,\beta})]\) where
\[
b(\alpha, \beta, r_{a,\beta}) = \frac{m(\alpha, \beta)}{r_{a,\beta} + \sqrt{r_{a,\beta} + m(\alpha, \beta)}}. \tag{5.40}
\]

Fix \( \omega \in (0, r_{a,\beta} - T(\alpha, \beta)) \) and \( \mu = r_{a,\beta} - \omega \). Then \( T(\alpha, \beta) < \mu < r_{a,\beta} \) and by Lemma 5.3
\[
0 < u(\alpha, \beta, \mu) \leq u(\alpha, \beta, r) \leq \alpha, \quad 0 \leq r \leq \mu. \tag{5.41}
\]
Since the system (2.2) is Lipschitz continuous in \( u \) and \( v \) when \( u \) is in a neighborhood of the interval \([u(\alpha, \beta, \mu), \alpha]\) in \((0, \infty)\), the continuous dependence on initial data implies that there exists \( \eta > 0 \) such that when \( |\gamma - \beta| < \eta \) the solution \( u(\alpha, \gamma, \cdot), v(\alpha, \gamma, \cdot) \) is defined on \([0, \mu]\) and \( u(\alpha, \gamma, r) > 0 \) for \( r \in [0, \mu] \), \( u'(\alpha, \gamma, r) < 0 \) for \( r \in (0, \mu) \), hence \( r_{a,\gamma} > \mu \). By taking \( \eta \) smaller if necessary, we can assume that \( T(\alpha, \gamma) < \mu \), hence \( T(\alpha, \gamma) < \mu < r_{a,\gamma} \). By Lemma 5.5 we can extend \( u(\alpha, \gamma, \cdot), v(\alpha, \gamma, \cdot) \) to \([0, \mu + b(\alpha, \gamma, \mu)]\). By taking \( \eta \) smaller if necessary, we can assume that
\[
b(\alpha, \gamma, \mu) > \frac{b(\alpha, \beta, \mu)}{2} > \frac{b(\alpha, \beta, r_{a,\beta})}{2} = 2c. \tag{5.42}
\]
Thus if we choose \( \omega \) to satisfy also \( \omega \leq c \), we get
\[
\mu + b(\alpha, \gamma, \mu) = r_{a,\beta} - \omega + b(\alpha, \gamma, \mu) \geq r_{a,\beta} + c. \tag{5.43}
\]
Thus \( u(\alpha, \gamma, \cdot), v(\alpha, \gamma, \cdot) \) extend to the interval \([0, r_{a,\beta} + c]\) and \( c < b(\alpha, \beta, r_{a,\beta}) \) so that \( u(\alpha, \beta, \cdot), v(\alpha, \beta, \cdot) \) also exist on \([0, r_{a,\beta} + c]\).

**Step 2.** We claim that there exist \( \varepsilon \in (0, c) \) and \( \delta \in (0, \eta) \) such that
\[
|u'(\alpha, \gamma, r) - u'(\alpha, \beta, r_{a,\beta})| \leq \frac{1}{2} |u'(\alpha, \beta, r_{a,\beta})| \tag{5.44}
\]
(recall that \( u'(\alpha, \beta, r_{a,\beta}) < 0 \)) when \( |\gamma - \beta| < \delta \) and \( |r - r_{a,\beta}| \leq \varepsilon \). Let \( \varepsilon \in (0, c) \), \( |\gamma - \beta| < \eta \), and \( r \in [r_{a,\beta} - \varepsilon, r_{a,\beta} + \varepsilon] \). By Step 1 and integration of (2.2) we have
\[
u'(\alpha, \gamma, r) - u'(\alpha, \beta, r_{a,\beta})
= u'(\alpha, \gamma, r) - u'(\alpha, \beta, r) + u'(\alpha, \beta, r) - u'(\alpha, \beta, r_{a,\beta})
= (u'(\alpha, \gamma, r_{a,\beta} - \varepsilon) - u'(\alpha, \beta, r_{a,\beta} - \varepsilon)) \left(\frac{r_{a,\beta} - \varepsilon}{r_{a,\beta}}\right)^{n-1}
+ \int_{r_{a,\beta} - \varepsilon}^{r} s^{n-1} \left(\frac{v(\alpha, \gamma, s) - q^{-1} v(\alpha, \gamma, s)}{q^{-1} v(\alpha, \gamma, s)}\right) ds
+ u'(\alpha, \beta, r_{a,\beta}) \left(\frac{r_{a,\beta}^{n-1} - 1}{r_{a,\beta}^{n-1}}\right)
+ \int_{r_{a,\beta}}^{r} s^{n-1} \left(\frac{v(\alpha, \beta, s) - q^{-1} v(\alpha, \beta, s)}{q^{-1} v(\alpha, \beta, s)}\right) ds. \tag{5.45}
\]
We deduce that

\[ \left| u'(\alpha, y, r) - u'(\alpha, \beta, r_{a, \beta}) \right| \]

\[ \leq \left| u'(\alpha, y, r_{a, \beta} - \varepsilon) - u'(\alpha, \beta, r_{a, \beta} - \varepsilon) \right| + \left| u'(\alpha, \beta, r_{a, \beta}) \right| - \frac{1}{r_{a, \beta} - 1} \]

\[ \leq \frac{1}{8} \left( r_{a, \beta} - 1 \right) \]  \hspace{1cm} (5.46)

and our claim follows.

The proof of Lemma 5.5 gives the following estimate for \( |\gamma - \beta| < \eta \):

\[ |v(\alpha, y, r)| \leq 2d \max \left( \gamma, |(p+1)/(q+1)| \right), \quad r_{a, \beta} - \varepsilon \leq r \leq r_{a, \beta} + \varepsilon. \]  \hspace{1cm} (5.47)

By making \( \varepsilon \) smaller if necessary we have

\[ \int_{r_{a, \beta} - \varepsilon}^{r_{a, \beta}} |v(\alpha, y, s)|^q ds + \int_{r_{a, \beta} - \varepsilon}^{r_{a, \beta}} |v(\alpha, \beta, s)|^q ds \leq \frac{1}{4} \left| u'(\alpha, \beta, r_{a, \beta}) \right|, \]

\[ \left| - \frac{1}{r_{a, \beta} - 1} \right| \leq \frac{1}{8} \]  \hspace{1cm} (5.48)

for \( r_{a, \beta} - \varepsilon \leq r \leq r_{a, \beta} + \varepsilon. \) Then from (5.46) we obtain

\[ |u'(\alpha, y, r) - u'(\alpha, \beta, r_{a, \beta})| \leq |u'(\alpha, y, r_{a, \beta} - \varepsilon) - u'(\alpha, \beta, r_{a, \beta} - \varepsilon)| + \frac{3}{8} |u'(\alpha, \beta, r_{a, \beta})| \]

\[ \leq \frac{1}{8} |u'(\alpha, \beta, r_{a, \beta})| \]  \hspace{1cm} (5.49)

for \( |\gamma - \beta| < \eta \) and \( |r - r_{a, \beta}| \leq \varepsilon. \) Now let \( \varepsilon \) be fixed. By continuous dependence on initial data and the fact that \( u(\alpha, \beta, r) > u(\alpha, \beta, r_{a, \beta} - \varepsilon) \) for \( r \in [0, r_{a, \beta} - \varepsilon) \), we can choose \( \delta \in (0, \eta) \) such that

\[ |u'(\alpha, y, r_{a, \beta} - \varepsilon) - u'(\alpha, \beta, r_{a, \beta} - \varepsilon)| \leq \frac{1}{8} |u'(\alpha, \beta, r_{a, \beta})| \]  \hspace{1cm} (5.50)

for \( |\gamma - \beta| < \delta \) and our claim follows.

Now assume that \( B \) is not open. Equation (5.18) implies that there exist \( \beta \in B \) and a sequence \( \beta_j \) in \( C \) such that \( \beta_j \to \beta \) and \( r_{a, \beta_j} \to T \in [0, \infty] \). Assume first that \( T > r_{a, \beta} \). Then we can assume that there exists \( \varepsilon' \in (0, \varepsilon) \) such that \( r_{a, \beta_j} \geq r_{a, \beta} + \varepsilon' \) for all \( j \). We can also assume that \( \varepsilon \) in Step 2 is such that \( 0 < \varepsilon < \varepsilon' \). Since \( u(\alpha, \beta, r_{a, \beta}) = 0 \) and \( u'(\alpha, \beta, r_{a, \beta}) < 0 \), there exists \( 0 < \varepsilon' \leq \varepsilon \) such that

\[ 0 < u(\alpha, \beta, r_{a, \beta} - \varepsilon') < \frac{1}{4} |u'(\alpha, \beta, r_{a, \beta})| \varepsilon. \]  \hspace{1cm} (5.51)

By continuous dependence on initial data, there exists \( \delta' \in (0, \delta) \) such that

\[ u(\alpha, y, r_{a, \beta} - \varepsilon') < 2u(\alpha, \beta, r_{a, \beta} - \varepsilon') \]  \hspace{1cm} (5.52)
Then we obtain a contradiction since \( j \) for \( j \geq j_0 \). By Step 2, for \( |r - r_{a,j}| \leq \varepsilon \) and \( j \geq j_0 \) we have

\[
|u'(\alpha, \beta, j, r)| = |u'(\alpha, \beta, r_{a,j})| + u'(\alpha, \beta, r_{a,j}) - u'(\alpha, \beta, j, r) \geq \frac{1}{2} |u'(\alpha, \beta, r_{a,j})|.
\] (5.53)

Therefore for \( j \geq j_0 \),

\[
u(\alpha, \beta, r_{a,j} + \varepsilon) \leq u(\alpha, \beta, r_{a,j} - \varepsilon') - \min_{|r - r_{a,j}| \leq \varepsilon} |u'(\alpha, \beta, j, r)| (\varepsilon + \varepsilon') < 2u(\alpha, \beta, r_{a,j} - \varepsilon') - \frac{1}{2} |u'(\alpha, \beta, r_{a,j})| \varepsilon < 0.
\] (5.54)

Then we obtain a contradiction since \( \beta_j \in C \). Now assume that \( T \leq r_{a,j} \). By Step 2 we have

\[
|u'(\alpha, \beta, j, r_{a,j}) - u'(\alpha, \beta, r_{a,j})| = |u'(\alpha, \beta, r_{a,j})| \leq \frac{1}{2} |u'(\alpha, \beta, r_{a,j})| = \frac{1}{2} |u'(\alpha, \beta, r_{a,j})|
\] (5.55)

for \( j \geq j_0 \) and we get a contradiction.

Now we can complete the proof of Theorem 1.3.

(i) Let \( \alpha > 0 \) be fixed. By Proposition 5.7, there exists a unique \( \beta > 0 \) such that \( u(\alpha, \beta, r_{a,j}) = u'(\alpha, \beta, r_{a,j}) = 0 \). With \( s \) and \( t \) defined in (2.1), we set

\[
w(r) = \left( \frac{r_{a,j}}{R} \right)^s u(\alpha, \beta, r_{a,j}/r), \quad z(r) = \left( \frac{r_{a,j}}{R} \right)^t v(\alpha, \beta, r_{a,j}/r), \quad 0 \leq r \leq R.
\] (5.56)

Then \((w, z)\) is a nontrivial radial solution of problem (1.1).

(ii) follows from Proposition 5.7.

\(\square\)

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References


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