INEQUALITY FOR RICCI CURVATURE OF CERTAIN
SUBMANIFOLDS IN LOCALLY CONFORMAL
ALMOST COSYMPLECTIC MANIFOLDS

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We establish inequalities between the Ricci curvature and the squared mean curvature, and also between the $k$-Ricci curvature and the scalar curvature for a slant, semi-slant, and bi-slant submanifold in a locally conformal almost cosymplectic manifold with arbitrary codimension.

1. Preliminaries

Let $\tilde{M}$ be a $(2m + 1)$-dimensional almost contact manifold with almost contact structure $(\varphi, \xi, \eta)$, that is, a global vector field $\xi$, a $(1,1)$ tensor field $\varphi$, and a 1-form $\eta$ on $\tilde{M}$ such that $\varphi^2 X = -X + \eta(X)\xi$, $\eta(\xi) = 1$ for any vector field $X$ on $\tilde{M}$. We consider a product manifold $\tilde{M} \times \mathbb{R}$, where $\mathbb{R}$ denotes a real line. Then a vector field on $\tilde{M} \times \mathbb{R}$ is given by $(X, f(d/dt))$, where $X$ is a vector field tangent to $\tilde{M}$, $t$ the coordinate of $\mathbb{R}$, and $f$ a function on $\tilde{M} \times \mathbb{R}$. We define a linear map $J$ on the tangent space of $\tilde{M} \times \mathbb{R}$ by $J(X, f(d/dt)) = (\varphi X - f\xi, \eta(X)(d/dt))$. Then we have $J^2 = -I$, and hence $J$ is an almost complex structure on $\tilde{M} \times \mathbb{R}$. The manifold $\tilde{M}$ is said to be normal (see [6]) if the almost complex structure $J$ is integrable (i.e., $J$ arises from a complex structure on $\tilde{M} \times \mathbb{R}$). Let $g$ be a Riemannian metric on $\tilde{M}$ compatible with $(\varphi, \xi, \eta)$, that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields $X$ and $Y$ tangent to $\tilde{M}$. Thus, the manifold $\tilde{M}$ is almost contact metric, and $(\varphi, \xi, \eta, g)$ is its almost contact metric structure. Clearly, we have $\eta(X) = g(X, \xi)$ for any vector field $X$ tangent to $\tilde{M}$. Let $\Phi$ denote the fundamental 2-form of $\tilde{M}$ defined by $\Phi(X, Y) = g(\varphi X, Y)$ for any vector fields $X$ and $Y$ tangent to $\tilde{M}$. The manifold $\tilde{M}$ is said to be almost cosymplectic if the forms $\eta$ and $\Phi$ are closed. That is, $d\eta = 0$ and $d\Phi = 0$, where $d$ is the operator of exterior differentiation. If $\tilde{M}$ is almost cosymplectic and normal, then it is called cosymplectic (see [1]). It is well known that the almost contact metric manifold is cosymplectic if and only if $\tilde{\nabla}\varphi$ vanishes identically, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. An almost contact metric manifold $\tilde{M}$ is a locally conformal almost cosymplectic manifold if and only if there exists a 1-form $\omega$ such that $d\Phi = 2\omega \wedge \Phi$, $d\eta = \omega \wedge \eta$, and $d\omega = 0$.

On the other hand, it is well-known that the Riemannian curvature tensor $\tilde{R}$ on a locally conformal almost cosymplectic manifold $\tilde{M}$ ($m \geq 2$) of pointwise constant $\varphi$-sectional
curvature $c$ satisfies (see[6])

$$g(\tilde{R}(X,Y)Z,W) = \frac{c - 3f^2}{4} \{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}$$

$$+ \frac{c + f^2}{4} \{g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) - 2g(X,\varphi Y)g(Z,\varphi W)\}$$

$$- \left(\frac{c + f^2}{4} + f'\right) \{g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(Y)\eta(W) + g(Y,Z)\eta(X)\eta(W)$$

$$- g(Y,W)\eta(X)\eta(Z)\} \quad X,Y,Z,W \in T_pM,$$

(1.1)

where $f$ is the function such that $\omega = f \eta$, $f' = \xi f$.

In [5], Lotta has introduced the following notion of slant submanifolds into almost contact metric manifolds. A submanifold $M$ tangent to $\xi$ in locally conformal almost cosymplectic manifold $\tilde{M}$ is said to be **slant** if for any $p \in M$ and any $X \in T_pM$, linearly independent of $\xi$, the angle between $\varphi X$ and $T_pM$ is a constant $\theta \in [0,\pi/2]$, called the **slant angle** of $M$ in $\tilde{M}$. Invariant and anti-invariant submanifolds of $\tilde{M}$ are slant submanifolds with slant angles $\theta = 0$ and $\theta = \pi/2$, respectively.

We say that a submanifold $M$ tangent to $\xi$ is a **bi-slant** submanifold in $\tilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on $M$ such that

1. $TM$ admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$;
2. for any $i = 1,2$, $\mathcal{D}_i$ is slant distribution with slant angle $\theta_i$.

On the other hand, CR-submanifolds of $\tilde{M}$ are bi-slant submanifolds with $\theta_1 = 0$, $\theta_2 = \pi/2$.

Let $2d_1 = \dim \mathcal{D}_1$ and $2d_2 = \dim \mathcal{D}_2$.

**Remark 1.1.** If either $d_1$ or $d_2$ vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds are particular cases of bi-slant submanifolds.

A submanifold $M$ tangent to $\xi$ is called a **semi-slant** submanifold in $\tilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on $M$ such that

1. $TM$ admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$;
2. the distribution $\mathcal{D}_1$ is an invariant distribution, that is, $\varphi(\mathcal{D}_1) = \mathcal{D}_1$;
3. the distribution $\mathcal{D}_2$ is slant with angle $\theta \neq 0$.

**Remark 1.2.** The invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that, in fact, semi-slant submanifolds are particular cases of bi-slant submanifolds.

1. If $d_2 = 0$, then $M$ is an invariant submanifold.
2. If $d_1 = 0$ and $\theta = \pi/2$, then $M$ is an anti-invariant submanifold.

For the other properties and examples of slant, bi-slant, and semi-slant submanifolds in an almost contact metric manifold, we refer to [2, 3].

Let $M$ be an $n$-dimensional submanifold of a locally conformal almost cosymplectic manifold $\tilde{M}$ equipped with a Riemannian metric $g$. The Gauss and Weingarten formulas
are given, respectively, by
\[ \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X N = -ANX + \nabla^\bot_X N, \] (1.2)
for all \( X, Y \in TM \) and \( N \in T^\bot M \), where \( \tilde{\nabla}, \nabla \), and \( \nabla^\bot \) are the Riemannian, induced Riemannian, and induced normal connections in \( \tilde{\mathcal{M}}, \mathcal{M} \), and the normal bundle \( T^\bot M \) of \( M \), respectively, and \( h \) is the second fundamental form related to the shape operator \( A \) by \( g(h(X, Y), N) = g(ANX, Y) \). Also, let \( R \) be the Riemannian curvature tensor of \( M \). Then the equation of Gauss is given by
\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \] (1.3)
for any vectors \( X, Y, Z, W \) tangent to \( M \).

For any vector \( X \) tangent to \( M \), we put \( \varphi X = PX + FX \), where \( PX \) and \( FX \) are the tangential and the normal components of \( \varphi X \), respectively. Given an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( M \), we define the squared norm of \( P \) by
\[ \|P\|^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j) \] (1.4)
and the mean curvature vector \( H(p) \) at \( p \in M \) is given by \( H = (1/n) \sum_{i=1}^n h(e_i, e_i) \).

We put
\[ h_{ij}^r = g(h(e_i, e_j), e_r), \quad \|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)), \] (1.5)
where \( \{e_{n+1}, \ldots, e_{2m+1}\} \) is an orthonormal basis of \( T^\bot_p M \) and \( r = n + 1, \ldots, 2m + 1 \). A submanifold \( M \) in \( \tilde{\mathcal{M}} \) is called totally geodesic if the second fundamental form vanishes identically and totally umbilical if there is a real number \( \lambda \) such that \( h(X, Y) = \lambda g(X, Y)H \) for any tangent vectors \( X, Y \) on \( M \).

For an \( n \)-dimensional Riemannian manifold \( M \), we denote by \( K(\pi) \) the sectional curvature of \( M \) associated with a plane section \( \pi \subset T_p M, p \in M \). For an orthonormal basis \( \{e_1, \ldots, e_n\} \) of the tangent space \( T_p M \), the scalar curvature \( \tau \) is defined by
\[ \tau = \sum_{i<j} K_{ij}, \] (1.6)
where \( K_{ij} \) denotes the sectional curvature of the 2-plane section spanned by \( e_i \) and \( e_j \).

Suppose that \( L \) is a \( k \)-plane section of \( T_p M \) and \( X \) a unit vector in \( L \). We choose an orthonormal basis \( \{e_1, \ldots, e_k\} \) of \( L \) such that \( e_1 = X \). Define the Ricci curvature \( \text{Ric}_L \) of \( L \) at \( X \) by
\[ \text{Ric}_L(X) = K_{12} + \cdots + K_{1k}. \] (1.7)
We simply called such a curvature a \( k \)-Ricci curvature. The scalar curvature \( \tau \) of the \( k \)-plane section \( L \) is given by

\[
\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}. \tag{1.8}
\]

For each integer \( k, 2 \leq k \leq n \), the Riemannian invariant \( \Theta_k \) on an \( n \)-dimensional Riemannian manifold \( M \) is defined by

\[
\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} \text{Ric}_L(X), \quad p \in M, \tag{1.9}
\]

where \( L \) runs over all \( k \)-plane sections in \( T_p M \) and \( X \) runs over all unit vectors in \( L \).

Recall that for a submanifold \( M \) in a Riemannian manifold, the relative null space of \( M \) at a point \( p \in M \) is defined by

\[
N_p = \{ X \in T_p M \mid h(X,Y) = 0 \ \forall \ Y \in T_p M \}. \tag{1.10}
\]

2. Ricci curvature and squared mean curvature

Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [4]). We prove similar inequalities for slant, bi-slant, and semi-slant submanifolds in a locally conformal almost cosymplectic manifold \( \widetilde{M} \). We consider submanifolds \( M \) tangent to \( \xi \).

**Theorem 2.1.** Let \( M \) be an \( n \)-dimensional \( \theta \)-slant submanifold tangent to \( \xi \) into a \((2m+1)\)-dimensional locally conformal almost cosymplectic manifold \( \widetilde{M} \). Then, the following hold.

1. For each unit vector \( X \in T_p M \) orthogonal to \( \xi \),

\[
\text{Ric}(X) \leq \frac{1}{4} \left\{ (n-1)(c-3f^2) + \frac{3}{2} (c+f^2) \cos^2 \theta - 4 \left( \frac{c+f^2}{4} + f' \right) + n^2 \|H\|^2 \right\}. \tag{2.1}
\]

2. If \( H(p) = 0 \), then a unit tangent vector \( X \) orthogonal to \( \xi \) at \( p \) satisfies the equality case of (2.1) if and only if \( X \in N_p \).

3. The equality case of (2.1) holds identically for all unit tangent vectors orthogonal to \( \xi \) at \( p \) if and only if \( p \) is a totally geodesic point.

**Proof.** (1) Let \( X \in T_p M \) be a unit tangent vector at \( p \) orthogonal to \( \xi \). We choose an orthonormal basis \( e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1} \), such that \( e_1, \ldots, e_n \) are tangent to \( M \) at \( p \) with \( e_1 = X \). Then, from the equation of Gauss, we have

\[
n^2 \|H\|^2 = 2\tau + \|h\|^2 - \frac{n(n-1)(c-3f^2)}{4} - \frac{3(n-1)(c+f^2)}{4} \cos^2 \theta + 2(n-1) \left( \frac{c+f^2}{4} + f' \right). \tag{2.2}
\]
From (2.2), we get

\[
\begin{align*}
n^2 \|H\|^2 &= 2\tau + \sum_{r=1}^{2m+1} \left[ (h'_{11})^2 + (h'_{22} + \cdots + h'_{nn})^2 + 2 \sum_{1 \leq i < j \leq n} (h'_{ij})^2 \right] \\
&- 2 \sum_{r=1}^{2m+1} \sum_{2 \leq i < j \leq n} h'_{ii} h'_{jj} - \frac{n(n-1)(c-3 f^2)}{4} \\
&- \frac{3(n-1)(c+f^2)}{4} \cos^2 \theta + 2(n-1) \left( \frac{c+f^2}{4} + f' \right) \\
&= 2\tau + \frac{1}{2} \sum_{r=1}^{2m+1} \left[ (h'_{11} + h'_{22} + \cdots + h'_{nn})^2 + (h'_{11} - h'_{22} - \cdots - h'_{nn})^2 \right] \\
&+ 2 \sum_{r=2m+2}^{2m+1} \sum_{1 \leq i < j \leq n} (h_{ij})^2 - 2 \sum_{r=2m+2}^{2m+1} \sum_{1 \leq i < j \leq n} h_{ii} h_{jj} \\
&- \frac{n(n-1)(c-3 f^2)}{4} - \frac{3(n-1)(c+f^2)}{4} \cos^2 \theta + 2(n-1) \left( \frac{c+f^2}{4} + f' \right).
\end{align*}
\]

(2.3)

By using the equation of Gauss, we have

\[
\sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=2m+2}^{2m+1} \sum_{1 \leq i < j \leq n} \left[ h'_{ii} h'_{jj} - (h_{ij})^2 \right] + \frac{(n-1)(n-2)(c-3 f^2)}{8} \\
+ \frac{3(n-2)(c+f^2)}{8} \cos^2 \theta + \frac{1}{2} \left( \frac{c+f^2}{4} + f' \right)(-2n+4). \tag{2.4}
\]

Substituting (2.4) in (2.3), we get

\[
\frac{1}{2} n^2 \|H\|^2 \geq 2 \text{Ric}(X) - \frac{(n-1)(c-3 f^2)}{2} - \frac{3(c+f^2)}{4} \cos^2 \theta + 2 \left( \frac{c+f^2}{4} + f' \right), \tag{2.5}
\]

or equivalently (2.1).

(2) Assume that \(H(P) = 0\). Equality holds in (2.1) if and only if

\[
\begin{align*}
h'_{12} &= \cdots = h'_{1n} = 0, \\
h'_{11} &= h'_{22} + \cdots + h'_{nn}, \quad r \in \{n+1, \ldots, 2m+1\}.
\end{align*} \tag{2.6}
\]

Then \(h'_{ij} = 0\) for all \(j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m+1\}\), that is, \(X \in N_p\).

(3) Then equality case of (2.1) holds for all unit tangent vectors orthogonal to \(\xi\) at \(p\) if and only if

\[
\begin{align*}
h'_{ij} &= 0, \quad i \neq j, r \in \{n+1, \ldots, 2m+1\}, \\
h'_{11} + \cdots + h'_{nn} &= 2h'_{ii} = 0, \quad i \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m+1\}.
\end{align*} \tag{2.7}
\]

In this case, it follows that \(p\) is a totally geodesic point. The converse is trivial. \(\square\)
Theorem 2.2. Let $M$ be an $n$-dimensional bi-slant submanifold satisfying $g(X, \varphi Y) = 0$, for any $X \in \mathcal{D}_1$ and any $Y \in \mathcal{D}_2$, tangent to $\xi$ in a $(2m + 1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}$. Then, the following hold.

(1) For each unit vector $X \in T_pM$ orthogonal to $\xi$ and if

(i) $X$ is tangent to $\mathcal{D}_1$,

$$\text{Ric}(X) \leq \frac{1}{4} \left\{ (n - 1)(c - 3f^2) + \frac{3}{2}(c + f^2) \cos^2 \theta_1 - 4 \left( \frac{c + f^2}{4} + f' \right) + n^2 \|H\|^2 \right\},$$

and if

(ii) $X$ is tangent to $\mathcal{D}_2$,

$$\text{Ric}(X) \leq \frac{1}{4} \left\{ (n - 1)(c - 3f^2) + \frac{3}{2}(c + f^2) \cos^2 \theta_2 - 4 \left( \frac{c + f^2}{4} + f' \right) + n^2 \|H\|^2 \right\}. \tag{2.8}$$

(2) If $H(p) = 0$, then a unit tangent vector $X$ orthogonal to $\xi$ at $p$ satisfies the equality case of (2.8) and (2.9) if and only if $X \in N_p$.

(3) The equality case of (2.8) and (2.9) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

Proof. (1) Let $X \in T_pM$ be a unit tangent vector at $p$ orthogonal to $\xi$. We choose an orthonormal basis $e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}$ such that $e_1, \ldots, e_n$ are tangent to $M$ at $p$ with $e_1 = X$. Then, from the equation of Gauss, we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - \frac{n(n - 1)(c - 3f^2)}{4} - \frac{6(c + f^2)}{4} \left( d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2 \right) + 2(n - 1) \left( \frac{c + f^2}{4} + f' \right), \tag{2.10}$$

where $2d_1 = \dim \mathcal{D}_1$ and $2d_2 = \dim \mathcal{D}_2$.

From (2.10), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{r = n+1}^{2m+1} \left[ (h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2 \sum_{1 \leq i < j \leq n} (h_{ij}^r)^2 \right] - \frac{2}{n+1} \sum_{1 \leq i < j \leq n} \sum_{r = n+1}^{2m+1} h_{ij}^r h_{ji}^r - \frac{n(n - 1)(c - 3f^2)}{4} - \frac{6(c + f^2)}{4} \left( d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2 \right) + 2(n - 1) \left( \frac{c + f^2}{4} + f' \right)$$
\[2 \tau + \frac{1}{2} \sum_{r=1}^{2m+1} \left( (h'_{11} + h'_{22} + \cdots + h'_{nn})^2 + (h'_{11} - h'_{22} - \cdots - h'_{nn})^2 \right) + 2 \sum_{r=1}^{2m+1} \frac{1}{2} \sum_{1 \leq i < j \leq n} (h'_{ij})^2 - 2 \sum_{r=1}^{2m+1} \sum_{1 \leq i < j \leq n} h'_{ii} h'_{jj} - \frac{n(n-1)(c-3f^2)}{4} \]

\[-\frac{6(c + f^2)}{4} (d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) + 2(n-1) \left( \frac{c + f^2}{4} + f' \right) \cdot (2.11)\]

We distinguish two cases.

(i) If \( X \) is tangent to \( \mathbb{D}_1 \), then we have

\[ \sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=1}^{2m+1} \sum_{1 \leq i < j \leq n} \frac{(n-1)(n-2)(c-3f^2)}{8} + \frac{c + f^2}{8} \left( 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 3 \cos^2 \theta_1 \right) + \frac{1}{2} \left( \frac{c + f^2}{4} + f' \right) (-2n+4). \]

(2.12)

Substituting (2.12) in (2.11), one gets

\[ \frac{1}{2} n^2 \|H\|^2 \geq 2 \text{Ric}(X) - \frac{(n-1)(c-3f^2)}{2} - \frac{3(c + f^2)}{4} \cos^2 \theta_1 + 2 \left( \frac{c + f^2}{4} + f' \right) \cdot (2.13) \]

which is equivalent to (2.8).

(ii) If \( X \) is tangent to \( \mathbb{D}_2 \), then we have

\[ \sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=1}^{2m+1} \sum_{1 \leq i < j \leq n} \frac{(n-1)(n-2)(c-3f^2)}{8} + \frac{c + f^2}{8} \left( 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 3 \cos^2 \theta_2 \right) + \frac{1}{2} \left( \frac{c + f^2}{4} + f' \right) (-2n+4). \]

(2.14)

Substituting (2.14) in (2.11), one gets

\[ \frac{1}{2} n^2 \|H\|^2 \geq 2 \text{Ric}(X) - \frac{(n-1)(c-3f^2)}{2} - \frac{3(c + f^2)}{4} \cos^2 \theta_2 + 2 \left( \frac{c + f^2}{4} + f' \right) \cdot (2.15) \]

which is equivalent to (2.9).

(2) Assume that \( H(p) = 0 \). Equality holds in (2.8) and (2.9) if and only if

\[ h'_{12} = \cdots = h'_{1n} = 0, \]

\[ h'_{11} = h'_{22} + \cdots + h'_{nn}, \quad r \in \{n+1, \ldots, 2m+1\}. \]

(2.16)

Then \( h'_{ij} = 0 \) for all \( j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m+1\} \), that is, \( X \in N_p \).
Then equality case of (2.8) and (2.9) holds for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if

$$h_{ij}' = 0, \quad i \neq j, \ r \in \{n + 1, \ldots, 2m + 1\},$$

$$h_{11}' + \cdots + h_{nn}' - 2h_{ii}' = 0, \quad i \in \{1, \ldots, n\}, \ r \in \{n + 1, \ldots, 2m + 1\}.$$

(2.17)

In this case, it follows that $p$ is a totally geodesic point. The converse is trivial. \hfill $\square$

**Corollary 2.3.** Let $M$ be an $n$-dimensional semi-slant submanifold in a $(2m + 1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}$. Then, the following hold.

1. For each unit vector $X \in T_pM$ orthogonal to $\xi$ and if
   
   (i) $X$ is tangent to $\mathcal{D}_1$,
   
   $$\text{Ric}(X) \leq \frac{1}{4} \left\{ (n - 1)(c - 3f^2) - 4 \left( \frac{c + f^2}{4} + f' \right) + n^2\|H\|^2 \right\},$$

   (2.18)

   and if

   (ii) $X$ is tangent to $\mathcal{D}_2$,

   $$\text{Ric}(X) \leq \frac{1}{4} \left\{ (n - 1)(c - 3f^2) + \frac{3}{2}(c + f^2) \cos^2 \theta - 4 \left( \frac{c + f^2}{4} + f' \right) + n^2\|H\|^2 \right\}. (2.19)$$

2. If $H(p) = 0$, then a unit tangent vector $X$ orthogonal to $\xi$ at $p$ satisfies the equality case of (2.18) and (2.19) if and only if $X \in N_p$.

3. The equality case of (2.18) and (2.19) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

**Corollary 2.4.** Let $M$ be an $n$-dimensional invariant submanifold in a $(2m + 1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then, the following hold.

1. For each unit vector $X \in T_pM$ orthogonal to $\xi$,

   $$\text{Ric}(X) \leq \frac{1}{4} \left\{ (n - 1)(c - 3f^2) + \frac{3}{2}(c + f^2) - 4 \left( \frac{c + f^2}{4} + f' \right) + n^2\|H\|^2 \right\}. (2.20)$$

2. If $H(p) = 0$, then a unit tangent vector $X$ orthogonal to $\xi$ at $p$ satisfies the equality case of (2.20) if and only if $X \in N_p$.

3. The equality case of (2.20) holds identically for all unit tangent vectors orthogonal to $\xi$ at $p$ if and only if $p$ is a totally geodesic point.

**Corollary 2.5.** Let $M$ be an $n$-dimensional anti-invariant submanifold in a $(2m + 1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then, the following hold.

1. For each unit vector $X \in T_pM$ orthogonal to $\xi$,

   $$\text{Ric}(X) \leq \frac{1}{4} \left\{ (n - 1)(c - 3f^2) - 4 \left( \frac{c + f^2}{4} + f' \right) + n^2\|H\|^2 \right\}. (2.21)$$

2. If $H(p) = 0$, then a unit tangent vector $X$ orthogonal to $\xi$ at $p$ satisfies the equality case of (2.21) if and only if $X \in N_p$. 
(3) The equality case of (2.21) holds identically for all unit tangent vectors orthogonal to \( \xi \) at \( p \) if and only if \( p \) is a totally geodesic point.

3. \( k \)-Ricci curvature and squared mean curvature

In this section, we prove relationship between the \( k \)-Ricci curvature and the squared mean curvature for slant, bi-slant, and semi-slant submanifolds in a locally conformal almost cosymplectic manifold \( \tilde{M} \). We state an inequality between the scalar curvature and the squared mean curvature for submanifolds \( M \) tangent to the vector field \( \xi \).

**Theorem 3.1.** Let \( M \) be an \( n \)-dimensional \( \theta \)-slant submanifold tangent to \( \xi \) into a \((2m+1)\)-dimensional locally conformal almost cosymplectic manifold \( \tilde{M} \). Then,

\[
\|H\|^{2} \geq \frac{2\tau}{n(n-1)} - \frac{1}{4n} \left[ n(c - 3f^{2}) + 3(c + f^{2}) \cos^{2}\theta - 8 \left( \frac{c + f^{2}}{4} + f' \right) \right],
\]

equality holding at a point \( p \in M \) if and only if \( p \) is a totally umbilical point.

**Proof.** Let \( p \) be a point of \( M \). We choose an orthonormal basis \( \{e_{1}, e_{2}, \ldots, e_{n} = \xi\} \) for the tangent space \( T_{p}M \) and \( \{e_{n+1}, \ldots, e_{2m+1}\} \) for the normal space \( T_{p}^{\perp}M \) at \( p \) such that the normal vector \( e_{n+1} \) is in the direction of the mean curvature vector and \( e_{1}, e_{2}, \ldots, e_{n} \) diagonalize the shape operator \( A_{n+1} \). Then, we have

\[
A_{n+1} = \begin{pmatrix}
a_{1} & 0 & 0 & \cdots & 0 \\
0 & a_{2} & 0 & \cdots & 0 \\
0 & 0 & a_{3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{n}
\end{pmatrix},
\]

\[
A_{r} = (h'_{ij}), \quad \sum_{i=1}^{n} h_{ij}^{r} = 0, \quad n + 2 \leq r \leq 2m + 1.
\]

From the equation of Gauss,

\[
n^{2}\|H\|^{2} = 2\tau + \sum_{i=1}^{n} a_{i}^{2} + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h'_{ij})^{2} - \frac{n(n-1)(c - 3f^{2})}{4} \\
- \frac{3(n-1)(c + f^{2})}{4} \cos^{2}\theta + 2(n - 1) \left( \frac{c + f^{2}}{4} + f' \right).
\]

On the other hand,

\[
\sum_{i<j} (a_{i} - a_{j})^{2} = (n - 1) \sum_{i=1}^{n} a_{i}^{2} - 2 \sum_{i<j} a_{i}a_{j}.
\]
Therefore, from the above equation, we have
\[ n^2 \| H \|^2 = \left( \sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2. \] (3.5)

Combining (3.3) and (3.5),
\[ n(n-1)\| H \|^2 \geq 2\tau + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \frac{n(n-1)(c-3f^2)}{4} \]
\[ \geq 3(n-1)(c+f^2) \cos^2 \theta + 2(n-1) \left( \frac{c+f^2}{4} + f' \right), \] (3.6)
which implies inequality (3.1). If the equality sign of (3.1) holds at a point \( p \in M \), then from (3.4) and (3.6) we get \( A_r = 0 \) \( (r = n+2, \ldots, 2m+1) \) and \( a_1 = \cdots = a_n \). Consequently, \( p \) is a totally umbilical point. The converse is trivial. □

**Theorem 3.2.** Let \( M \) be an \( n \)-dimensional bi-slant submanifold satisfying \( g(X, \varphi Y) = 0 \), for any \( X \in \mathcal{D}_1 \) and any \( Y \in \mathcal{D}_2 \), tangent to \( \xi \) into a \((2m+1)\)-dimensional locally conformal almost cosymplectic manifold \( \tilde{M} \). Then,
\[ \| H \|^2 \geq \frac{2\tau}{n(n-1)} - \frac{1}{4n(n-1)} \left[ n(n-1)(c-3f^2) + 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)(c+f^2) \right. \]
\[ - \left. 8(n-1) \left( \frac{c+f^2}{4} + f' \right) \right], \] (3.7)
where \( 2d_1 = \dim \mathcal{D}_1 \) and \( 2d_2 = \dim \mathcal{D}_2 \).

**Theorem 3.3.** Let \( M \) be an \( n \)-dimensional semi-slant submanifold tangent to \( \xi \) into a \((2m+1)\)-dimensional locally conformal almost cosymplectic manifold \( \tilde{M} \). Then,
\[ \| H \|^2 \geq \frac{2\tau}{n(n-1)} - \frac{1}{4n(n-1)} \left[ n(n-1)(c-3f^2) + 6(d_1 + d_2 \cos^2 \theta)(c+f^2) \right. \]
\[ - \left. 8(n-1) \left( \frac{c+f^2}{4} + f' \right) \right], \] (3.8)
where \( 2d_1 = \dim \mathcal{D}_1 \) and \( 2d_2 = \dim \mathcal{D}_2 \).

**Theorem 3.4.** Let \( M \) be an \( n \)-dimensional \( \theta \)-slant submanifold tangent to \( \xi \) into a \((2m+1)\)-dimensional locally conformal almost cosymplectic manifold \( \tilde{M} \). Then, for any integer \( k \) \( (2 \leq k \leq n) \) and any point \( p \in M \),
\[ \| H \|^2 \geq \Theta_k(p) - \frac{1}{4n} \left[ n(c-3f^2) + 3(c+f^2) \cos^2 \theta - 8 \left( \frac{c+f^2}{4} + f' \right) \right]. \] (3.9)
Proof. Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_pM \). Denote by \( L_{i_1 \cdots i_k} \) the \( k \)-plane section spanned by \( e_{i_1}, \ldots, e_{i_k} \). It follows from (1.7) and (1.8) that

\[
\tau(L_{i_1 \cdots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \text{Ric}_{L_{i_1 \cdots i_k}}(e_i),
\]

\[
\tau(p) = \frac{1}{\binom{n-2}{k-2}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \tau(L_{i_1 \cdots i_k}).
\]

(3.10)

Combining (1.9) and (3.10), we obtain

\[
\tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p).
\]

(3.11)

Therefore, by using (3.1) and (3.11), we can obtain the inequality in Theorem 3.4. □

Theorem 3.5. Let \( M \) be an \( n \)-dimensional bi-slant submanifold tangent to \( \xi \) into a \((2m+1)\)-dimensional locally conformal almost cosymplectic manifold \( \tilde{M} \). Then, for any integer \( k \) \((2 \leq k \leq n)\) and any point \( p \in M \),

\[
\|H\|^2 \geq \Theta_k(p) - \frac{1}{4n(n-1)} \left[ n(n-1)(c-3f^2) + 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) (c + f^2) - 8(n-1) \left( \frac{c + f^2}{4} + f' \right) \right],
\]

(3.12)

where \( 2d_1 = \dim D_1 \) and \( 2d_2 = \dim D_2 \).

Theorem 3.6. Let \( M \) be an \( n \)-dimensional semi-slant submanifold tangent to \( \xi \) into a \((2m+1)\)-dimensional locally conformal almost cosymplectic manifold \( \tilde{M} \). Then, for any integer \( k \) \((2 \leq k \leq n)\) and any point \( p \in M \),

\[
\|H\|^2 \geq \Theta_k(p) - \frac{1}{4n(n-1)} \left[ n(n-1)(c-3f^2) + 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) (c + f^2) - 8(n-1) \left( \frac{c + f^2}{4} + f' \right) \right],
\]

(3.13)

where \( 2d_1 = \dim D_1 \) and \( 2d_2 = \dim D_2 \).

Corollary 3.7. Let \( M \) be an \( n \)-dimensional invariant submanifold tangent to \( \xi \) into a \((2m+1)\)-dimensional locally conformal almost cosymplectic manifold \( \tilde{M} \). Then, for any integer \( k \) \((2 \leq k \leq n)\) and any point \( p \in M \),

\[
\|H\|^2 \geq \Theta_k(p) - \frac{1}{4n} \left[ n(c-3f^2) + 3(c + f^2) - 8 \left( \frac{c + f^2}{4} + f' \right) \right].
\]

(3.14)
Inequality for Ricci curvature

**Corollary 3.8.** Let $M$ be an $n$-dimensional anti-invariant submanifold tangent to $\xi$ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}$. Then, for any integer $k$ ($2 \leq k \leq n$) and any point $p \in M$,

$$\|H\|^2 \geq \Theta_k(p) - \frac{1}{4n}\left[n(c - 3f^2) - 8\left(\frac{c + f^2}{4} + f'\right)\right]. \quad (3.15)$$

**Corollary 3.9.** Let $M$ be an $n$-dimensional contact CR-submanifold tangent to $\xi$ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}$. Then, for any integer $k$ ($2 \leq k \leq n$) and any point $p \in M$,

$$\|H\|^2 \geq \Theta_k(p) - \frac{1}{4n(n-1)}\left[n(n-1)(c - 3f^2) + 6d_1(c + f^2) - 8(n-1)\left(\frac{c + f^2}{4} + f'\right)\right]. \quad (3.16)$$

**References**


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