A study is made of the propagation of time-harmonic plane thermoelastic waves of assigned frequency in an infinite rotating medium using Green-Naghdi model (1993) of linear thermoelasticity without energy dissipation. A more general dispersion equation is derived to examine the effect of rotation on the phase velocity of the modified coupled thermal dilatational shear waves. It is observed that in thermoelasticity theory of type II (Green-Naghdi model), the modified coupled dilatational thermal waves propagate unattenuated in contrast to the classical thermoelasticity theory, where the thermoelastic waves undergo attenuation (Parkus, Chadwick, and Sneddon). The solutions of the more general dispersion equation are obtained for small thermoelastic coupling by perturbation technique. Cases of high and low frequencies are also analyzed. The rotation of the medium affects both quasielastic dilatational and shear wave speeds to the first order in \( \omega \) for low frequency, while the quasithermal wave speed is affected by rotation up to the second power in \( \omega \). However, for large frequency, rotation influences both the quasi-dilatational and shear wave speeds to first order in \( \omega \) and the quasithermal wave speed to the second order in \( 1/\omega \).

1. Introduction

Study of plane thermoelastic and magnetothermoelastic wave propagation in a nonrotating medium is receiving considerable attention in recent years. The classical theory of thermoelasticity is based on Fourier’s law which predicts an infinite speed of heat propagation. In order to eliminate this paradox of infinite speed of thermal propagation, Lord and Shulman [8] employed a modified generalized thermoelastic theory which is hyperbolic in nature. The Lord-Shulman theory with a thermal relaxation time has been used by several authors, such as Puri [11] and Nayfeh and Nemat-Nasser [9], to study plane thermoelastic waves in nonrotating infinite media. In generalized thermoelasticity, Agarwal [1] has made an investigation of surface waves. Using Green-Lindsay theory [6], Agarwal [2, 3] studied, respectively, thermoelastic and magnetothermoelastic plane wave propagation in infinite nonrotating media. Schoenberg and Censor [15] studied the propagation of plane harmonic waves in a rotating elastic medium without thermal field.
It has been shown there that the rotation causes the elastic medium to be dispersive and anisotropic. This study included some discussion on the free surface phenomenon in a rotating half-space.

It appears that little attention has been paid to the study of propagation of plane thermoelastic waves in a rotating medium. Since most large bodies, like the earth, the moon, and other planets, have an angular velocity, it appears more realistic to study the propagation of plane thermoelastic or magnetothermoelastic waves in a rotating medium with thermal relaxation.


Recently, a theory of thermoelasticity (type II) without energy dissipation (Green-Naghdi model [7]) is proposed, where in the theory of generalized thermoelasticity possesses several significant characteristics that differ from the traditional classical development in thermoelastic material behavior: (i) it does not sustain energy dissipation, (ii) the entropy flux vector (or equivalently the heat flow vector) in the theory is determined in terms of the same potential that also determines the stress, (iii) it permits transmission of heat flow as thermal waves at finite speed.

In the present paper, Green-Naghdi model [7] of linear thermoelasticity without energy dissipation is used to investigate the propagation of harmonically time-dependent plane thermoelastic waves in an infinite rotating medium. A more general dispersion equation for propagation of coupled thermal dilatational shear waves without energy dissipation in a rotating medium incorporating the effect of rotation is obtained. The solutions of the more general dispersion equation are obtained for small thermoelastic coupling by a perturbation technique. Cases of low and high frequencies are also studied to examine the effects of rotation and the small thermoelastic coupling on the phase velocity of the waves. It is observed that in the thermoelasticity theory of type II (Green-Naghdi model), the modified coupled dilatational and shear waves propagate unattenuated in contrast to the classical thermoelasticity theory, where the thermoelastic waves undergo attenuation (see [4, 5, 10]). It is also observed that rotation of the medium affects shear waves. Rotation of the medium affects both quasidilatational and shear wave speeds to the first order in $\omega$ for low frequency, while the quasithermal wave speed is affected by rotation up to the second power in $\omega$. However, for large frequency, rotation affects both quasidilatational and shear wave speeds to the first order in $\omega$ and the quasithermal wave speed to the second order in $1/\omega$.

It may be mentioned that a similar problem of wave propagation in a thermoelastic rotating medium was studied by Roychoudhuri [12] using Green-Lindsay model of generalized thermoelasticity with two relaxation times.

2. Problem formulation and basic equations

An infinite isotropic, homogeneous, thermally conducting elastic medium with density $\rho$ and Lame' constants $\lambda$, $\mu$ is considered. The medium is rotating uniformly with angular velocity $\tilde{\Omega} = \Omega \hat{n}$, where $\hat{n}$ is a unit vector representing the direction of the axis of
rotation. The displacement equation of motion in the rotating frame of reference has two additional terms:

(i) centripetal acceleration $\hat{\Omega} \times (\hat{\Omega} \times \hat{u})$ due to the time-varying motion only;
(ii) the Coriolis acceleration $2\hat{\Omega} \times \dot{\hat{u}}$.

Here, $\hat{u}$ is the dynamic displacement vector measured from a steady-state deformed position and is supposed to be small. These two terms do not appear in the equations for nonrotating media.

The stress equations of motion, in the absence of body forces, are

$$\tau_{ij, j} = \rho \left[ \ddot{\vec{u}}_i + \{\hat{\Omega} \times (\hat{\Omega} \times \vec{u}) \}_i + (2\hat{\Omega} \times \dot{\vec{u}})_i \right], \quad (2.1)$$

where $\tau_{ij} = \lambda \Delta \delta_{ij} + 2 \mu e_{ij} - \gamma \theta \delta_{ij}$ with

$$2e_{ij} = u_{i,j} + u_{j,i}, \quad i, j = 1, 2, 3, \quad (2.2)$$

$\lambda, \mu$ are Lame’ constants, $\gamma = \alpha_t (3\lambda + 2\mu), \alpha_t$ is the coefficient of linear thermal expansion of the material, $\rho$ is the constant mass density, $\Delta$ is the dilatational, $\theta$ is the temperature above uniform reference temperature $\theta_0$, $\tau_{ij}$ is the stress tensor, $e_{ij}$ is the strain tensor, $u_i$ are the displacement components.

Combining (2.1) and (2.2), we obtain the displacement equation of motion in the rotating frame of reference as

$$\rho \left[ \ddot{\vec{u}} + \hat{\Omega} \times (\hat{\Omega} \times \vec{u}) \right]_i + 2\hat{\Omega} \times \dot{\vec{u}}_i = (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) + \mu \nabla^2 \vec{u} - \gamma \nabla \theta. \quad (2.3)$$

Again, linearized form of the theory of thermoelasticity [type II] without energy dissipation proposed by Green and Naghdi [7] consists of the following heat conduction equation:

$$\rho C_v \ddot{\theta} + \gamma \theta_0 \text{div} \vec{u} = K^* \nabla^2 \theta, \quad K^* > 0, \quad (2.4)$$

where $K^*$ is a material constant characteristic of the theory and $C_v$ is the specific heat. Using the dimensionless quantities

$$\omega = \frac{c_1}{g \omega^*} U_i, \quad t = \frac{\eta}{\omega^*}, \quad x_i = \frac{c_1}{\omega^*} \xi_i, \quad \theta = \theta_0 \Theta, \quad \Omega = \omega^* \Omega'$$

$$\omega^* = \frac{\rho C_v c_1^2}{K^*}, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad g = \frac{\gamma}{\rho C_v},$$

$$\beta^2 = \frac{\lambda + 2\mu}{\mu}, \quad b = \frac{\gamma \theta_0}{\mu}, \quad \varepsilon_{\theta} = \frac{bg}{\beta^2}, \quad (2.5)$$
and writing

\[
U_1 = U, \quad U_2 = V, \quad U_3 = W, \\
x_1 = x, \quad x_2 = y, \quad x_3 = z,
\]

(2.6)

(2.3) and (2.4) in nondimensional form reduce to

\[
\beta^2 \left[ \ddot{U}_i + \{\ddot{\Omega}' \times (\ddot{\Omega}' \times \ddot{U})\}_i + \{2\ddot{\Omega}' \times \ddot{U}\}_i \right] = (\beta^2 - 1) \Delta_i + \nabla^2 U_i - \beta^2 \epsilon\theta \Theta_i, \quad i = 1, 2, 3
\]

\[
\omega^* \ddot{\Theta} + \omega^* \text{div} \ddot{U} = \Theta_{,ii}.
\]

(2.7)

3. Plane harmonic waves in infinite rotating medium

We are concerned with isotropic solid medium, where we may consider waves propagating in the \(x_1\) direction. So, all the field variables are functions of the coordinate \(x_1\) and time \(t\). We concentrate our attention on the time-dependent stresses and displacement only that are caused by centripetal force and possible body forces. As such, we consider time-varying dynamic solutions and the time-independent part of the centripetal acceleration is neglected. To examine the effect of rotation and thermoelastic couplings on the coupled elastic dilatational, shear, and thermal waves, we set \(\hat{\Omega} = (0,0,\Omega)\), where \(\Omega\) is a constant. In view of these assumptions, (2.7) reduce to

\[
\ddot{U} - \Omega'^2 U - 2\Omega' \dot{V} = U'' - \epsilon\theta \Theta',
\]

(3.1)

\[
\beta^2 [ \ddot{V} - \Omega'^2 V + 2\Omega' \dot{U} ] = V'',
\]

(3.2)

\[
\omega^* \ddot{\Theta} - \Theta'' + \omega^* \ddot{U} = 0,
\]

(3.3)

\[
\beta^2 \ddot{W} = W''.
\]

(3.4)

Equations (3.1), (3.2), and (3.3) constitute a coupled system and represent coupled thermal dilatational shear waves, while (3.4) uncouples from the system. The thermal field affects shear motion due to rotation. This coupling disappears when rotation vanishes.

We set

\[
(U, V, \Theta) = (a_1, a_2, a_3) \exp \{i(qx - t\omega)\},
\]

(3.5)

where \(a_i\)'s \((i = 1, 2, 3)\) are all constants. Here, \(\omega\) is the prescribed frequency; \(q\) is the wave number to be determined. The phase velocity is \(c = \omega/\text{Re} q\) and attenuation coefficient is \(S = -\text{Im}(q)\).

Substituting (3.5) into (3.1), (3.2), and (3.3), we obtain

\[
(q^2 - \omega^2 - \Omega'^2) a_1 - 2i\omega \Omega' a_2 + iq\epsilon\theta a_3 = 0,
\]

\[
2i\omega \Omega' \beta^2 a_1 + (q^2 - \beta^2 \omega^2 - \beta^2 \Omega'^2) a_2 = 0,
\]

\[
-iaq^* \omega^2 a_1 + (q^2 - \omega^* \omega^2) a_3 = 0.
\]

(3.6)
For nontrivial solutions of the system (3.6), we arrive at the dispersion equation for the coupled waves as

\[
\begin{vmatrix}
q^2 - \omega^2 - \Omega^2 & -2i\Omega' & iq\varepsilon_\theta \\
2i\Omega' \omega \beta^2 & q^2 - \beta^2 \omega^2 - \beta^2 \Omega^2 & 0 \\
-iq\omega^* \omega^2 & 0 & q^2 - \omega^* \omega^2
\end{vmatrix} = 0.
\]

(3.7)

When expanded, (3.7) becomes

\[
(q^2 - \omega^* \omega^2)[(q^2 - \omega^2 - \Omega^2)(q^2 - \beta^2(\omega^2 + \Omega^2)) - 4\omega^2 \Omega^2 \beta^2]
- \varepsilon_\theta \omega^* \omega^2 q^2[q^2 - \beta^2(\omega^2 + \Omega^2)] = 0.
\]

(3.8)

If \(\Omega' = 0\), this reduces to

\[
(q^2 - \beta^2 \omega^2)[(q^2 - \omega^* \omega^2)(q^2 - \omega^2) - \varepsilon_\theta \omega^* \omega^2 q^2] = 0,
\]

(3.9)

leading to \((q^2 - \beta^2 \omega^2) = 0\), which corresponds to the uncoupled elastic shear wave in a nonrotating medium as expected and

\[
q^4 - \omega^2(1 + \omega^* + \varepsilon_\theta \omega^*) q^2 + \omega^* \omega^4 = 0.
\]

(3.10)

The roots of (3.10) are all real, showing that the thermoelastic waves do not undergo attenuation in thermoelasticity of type II without energy dissipation.

For \(\varepsilon_\theta = 0\),

\[
q_1 = \omega \quad \text{(taking +ve sign)}
\Rightarrow \frac{\omega}{\text{Re} q} = 1 \quad \text{(nondimensional dilatational wave speed)},
\]

\[
q_2 = \omega \sqrt{\omega^*} \quad \text{(taking -ve sign)}
\Rightarrow \frac{\omega}{\text{Re} q} = \frac{1}{\sqrt{\omega^*}} = \frac{\sqrt{K^*/\rho C_v}}{c_1} = \frac{C_T}{c_1},
\]

(3.11)

where \(C_T\) = finite thermal wave speed = \(\sqrt{K^*/\rho C_v}\) as is expected.

Equation (3.10) corresponds to the dispersion equation of the coupled thermal dilatational waves without energy dissipation in a thermoelastic nonrotating medium in contrast to the corresponding dispersion equation \(q^4 - q^2(\omega^2 - i\omega(1 + \varepsilon_\theta)) - i\omega^3 = 0\) for plane wave propagation in classical thermoelasticity [4, 5, 10], where waves undergo attenuation.

Thus, (3.8) is a more general dispersion equation for propagation of coupled thermal dilatational shear waves without energy dissipation in a rotating medium in the sense that it incorporates the effect of rotation and thermoelastic coupling \(\varepsilon_\theta\).
Thermoelastic wave propagation in a rotating elastic medium

To explore and delineate the thermal, dilatational, and shearing effects, we look for solutions for small $\varepsilon_\theta$.

Equation (3.8) for $\varepsilon_\theta = 0$ admits the following solutions: $q^2 = \omega^* \omega^2$,

$$2q_{2,1}^2 = (\beta^2 + 1) \Omega_0^2 \pm \left\{ (\beta^2 + 1)^2 \Omega_0^4 - 4\beta^2 (\omega^2 - \Omega^2) \right\}^{1/2} = 2J_{2,1}^2,$$

(3.12)

where $\Omega_0^2 = \omega^2 + \Omega^2$.

Since for $\Omega' = 0$, $q_1 = \omega$, $q_2 = \beta\omega$, we conclude that $q_1$ is the modified elastic dilatational wave speed and $q_2$ is the modified shear wave speed, both modified by rotation.

For small $\varepsilon_\theta$, we set

$$q^2 = q_u^2 = J_1^2 + \eta_u \varepsilon_\theta + O(\varepsilon_\theta)^2,$$

$$q^2 = q_v^2 = J_2^2 + \eta_v \varepsilon_\theta + O(\varepsilon_\theta)^2,$$

$$q^2 = q_\theta^2 = \omega^* \omega^2 + \eta_\theta \varepsilon_\theta + O(\varepsilon_\theta)^2.$$  

(3.13)

Substituting into (3.8) and equating coefficients of like powers of $\varepsilon_\theta$, we obtain

$$\eta_u = \left[ - \omega^* \omega^2 J_1^2 (J_1^2 - \beta^2 \omega^2 - \beta^2 \Omega^2) \right]$$

$$\times \left[ (J_1^2 - \omega^* \omega^2) (2J_1^2 - \omega^2 - \Omega^2 - \beta^2 \omega^2 - \beta^2 \Omega^2) \right. $$

$$\left. + (J_1^2 - \omega^2 - \Omega^2) (J_1^2 - \beta^2 \omega^2 - \beta^2 \Omega^2) - 4\beta^2 \omega^2 \Omega^2 \right]^{-1},$$

$$\eta_v = \left[ - \omega^* \omega^2 J_2^2 (J_2^2 - \beta^2 \omega^2 - \beta^2 \Omega^2) \right]$$

$$\times \left[ (J_2^2 - \omega^* \omega^2) (2J_2^2 - \omega^2 - \Omega^2 - \beta^2 \omega^2 - \beta^2 \Omega^2) \right. $$

$$\left. + (J_2^2 - \omega^2 - \Omega^2) (J_2^2 - \beta^2 \omega^2 - \beta^2 \Omega^2) - 4\beta^2 \omega^2 \Omega^2 \right]^{-1},$$

$$\eta_\theta = \left[ - \omega^* \omega^2 (\omega^* \omega^2 - \beta^2 \omega^2 - \beta^2 \Omega^2) \right]$$

$$\times \left[ (\omega^* \omega^2 - \omega^2 - \Omega^2) (\omega^* \omega^2 - \beta^2 \omega^2 - \beta^2 \Omega^2) - 4\beta^2 \omega^2 \Omega^2 \right]^{-1}.$$  

(3.14)

Following Puri [11] and Agarwal [2], we may call $q_u$, $q_v$, $q_\theta$ the speeds of modified quasielastic, quasishear, and quasithermal waves, respectively.

If $\Omega' = 0$,

$$\eta_u = \frac{-\omega^* \omega^2}{1 - \omega^*}, \quad \eta_v = 0, \quad \eta_\theta = \frac{-\omega^* \omega^2}{\omega^* - 1}. $$

(3.15)

Dilatational, shear, and thermal wave speed solutions (without rotation) then reduce to

$$q_u^2 = \omega^2 \left( 1 - \frac{\omega^* \varepsilon_\theta}{1 - \omega^*} \right), \quad q_v^2 = \beta^2 \omega^2, \quad q_\theta^2 = \omega^* \omega^2 \left( 1 - \frac{\omega^* \varepsilon_\theta}{\omega^* - 1} \right),$$

(3.16)

which are the results similar to those reported by Nayfeh and Nemat-Nasser [9] in the nonrotating case.
4. Case of large frequency

For large frequency $\omega$, 

$$
\eta_u \equiv K_1 + K_2 \frac{1}{\omega^2}, \quad \eta_v \equiv K'_1 + K_2 \frac{1}{\omega^2}, \quad \eta_\theta \equiv K_3 + K_4 \omega^2 + K_5 \frac{1}{\omega^2}, \quad (4.1)
$$

where

$$
K_1 = \frac{\omega^* f_1^2 \beta^2}{a_2}, \quad K'_1 = \frac{\omega^* f'_1^2 \beta^2}{a_2},
$$

$$
K_2 = \frac{\omega^* f_2^2 \beta^2}{a_2} \left( \frac{\beta^2 \Omega^2 f_1^2 - f_1^2}{\beta^2} - \frac{a_1 - 4 \beta^2 \Omega^2}{a_2} \right),
$$

$$
K'_2 = \frac{\omega^* f'_2^2 \beta^2}{a_2} \left( \frac{\beta^2 \Omega^2 f'_2^2 - f'_2^2}{\beta^2} - \frac{a'_1 - 4 \beta^2 \Omega^2}{a_2} \right),
$$

$$
K_3 = \frac{\omega^* (\omega^* - \beta^2)}{L_0} \left\{ \left( \frac{L_1 - 4 \beta^2 \Omega^2}{L_0} \right)^2 + \frac{\beta^2 \Omega^2 (L_1 - 4 \beta^2 \Omega^2)}{L_0 (\omega^* - \beta^2)} \right\},
$$

$$
K_4 = -\frac{\omega^* (\omega^* - \beta^2)}{L_0},
$$

$$
K_5 = -\frac{\omega^* (\omega^* - \beta^2)}{L_0} \left\{ \left( \frac{L_1 - 4 \beta^2 \Omega^2}{L_0} \right)^2 - \frac{\beta^2 \Omega^2 (L_1 - 4 \beta^2 \Omega^2)}{L_0 (\omega^* - \beta^2)} \right\},
$$

$$
a_1 = -f_1^2 (1 + \beta^2) - \omega^* (2 f_1^2 - \Omega^2 - \beta^2 \Omega^2) - \beta^2 (f_1^2 - \Omega^2) - (f_1^2 - \beta^2 \Omega^2),
$$

$$
a'_1 = -f'_2 (1 + \beta^2) - \omega^* (2 f'_2^2 - \Omega^2 - \beta^2 \Omega^2) - \beta^2 (f'_2^2 - \Omega^2) - (f'_2^2 - \beta^2 \Omega^2),
$$

$$
a_2 = \omega^* (1 + \beta^2) + \beta^2, \quad L_0 = (\omega^* - 1) (\omega^* - \beta^2),
$$

$$
L_1 = -\beta^2 \Omega^2 (\omega^* - 1) - \Omega^2 (\omega^* - \beta^2).
$$

Therefore,

$$
q''_u = f_1^2 + \left( K_1 + K_2 \frac{1}{\omega^2} \right) \epsilon_\theta,
$$

$$
q''_v = f'_2 + \left( K'_1 + K'_2 \frac{1}{\omega^2} \right) \epsilon_\theta,
$$

$$
q''_\theta = \omega^* \omega^2 + \left( K_3 + K_4 \omega^2 + K_5 \frac{1}{\omega^2} \right) \epsilon_\theta. \quad (4.3)
$$

Quasielastic dilatational wave speed modified by both rotation and thermoelastic coupling $\epsilon_\theta$ equals $\omega/q''_u \equiv (\omega/f_1) [1 - (1/2 f_1^2) (K_1 + K_2 (1/\omega^2))] \epsilon_\theta$.

Modified quasielastic shear wave speed $= \omega/q''_v \equiv (\omega/f'_2) [1 - (1/2 f'_2^2) (K'_1 + K'_2 (1/\omega^2))] \epsilon_\theta$.

Thus, rotation affects both quasidilatational and shear wave speeds to the first power in $1/\omega$ for large frequency, since $K_i, K'_i$ is independent of $\Omega'$. 
Thermoelastic wave propagation in a rotating elastic medium

Modified quasithermal wave speed \( \frac{\omega}{q_\theta} \equiv \frac{1}{\sqrt{\omega^*}} \left[ 1 - K_3 \epsilon_\theta^2/2\omega^* - K_4 \epsilon_\theta^2/2\omega^* \right] \). Rotation affects quasithermal wave speed to the second order in \( 1/\omega \) for large frequency.

5. Case of low frequency

For small \( \omega \),

\[
\eta_u \approx D_1 \omega^2, \quad \eta_v \approx D_2 \omega^2, \quad \eta_\theta \approx D_3 \omega^4, \quad (5.1)
\]

where

\[
D_1 = - \frac{\omega^* J_1^2 (J_1^2 - \beta^2 \Omega^2)}{a_0}, \quad D_2 = - \frac{\omega^* J_2^2 (J_2^2 - \beta^2 \Omega^2)}{a'_0}, \quad D_3 = \frac{\omega^*}{\Omega^2}, \quad (5.2)
\]

Therefore, \( q_u^2 = J_1^2 + D_1 \omega^2 \epsilon_\theta, \quad q_v^2 = J_2^2 + D_2 \omega^2 \epsilon_\theta, \quad q_\theta^2 = \omega^* \omega^2 + D_3 \omega^4 \epsilon_\theta \).

Modified dilatational wave speed \( \frac{\omega}{q_u} \equiv (\omega/J_1)[1 - D_1 \omega^2 \epsilon_\theta/2J_1^2] \), corresponding to the modified quasielastic dilatational wave speed influenced by rotation and thermal field.

Modified elastic shear wave speed \( \frac{\omega}{q_v} \equiv (\omega/J_2)[1 - D_2 \omega^2 \epsilon_\theta/2J_2^2] \), corresponding to the modified quasielastic shear wave influenced by rotation and thermal field.

Modified thermal wave speed \( \frac{\omega}{q_\theta} \equiv (1/\sqrt{\omega^*})[1 - D_3 \omega^2 \epsilon_\theta/2\omega^*] \), corresponding to the modified quasithermal wave influenced by rotation and thermoelastic coupling \( \epsilon_\theta \).

It is observed that the rotation of the medium affects both quasielastic dilatational and shear wave speeds to the first power in \( \omega \) for low frequency, while the thermal wave speed is affected by rotation up to the second power of \( \omega \).

References


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