INTERPOLATION METHODS TO ESTIMATE EIGENVALUE DISTRIBUTION OF SOME INTEGRAL OPERATORS

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We study the asymptotic distribution of eigenvalues of integral operators $T_k$ defined by kernels $k$ which belong to Triebel-Lizorkin function space $F^\sigma_{pu}(F^\tau_{qv})$ by using the factorization theorem and the Weyl numbers $x_n$. We use the relation between Triebel-Lizorkin space $F^\sigma_{pu}(\Omega)$ and Besov space $B^r_{pq}(\Omega)$ and the interpolation methods to get an estimation for the distribution of eigenvalues in Lizorkin spaces $F^\sigma_{pu}(F^\tau_{qv})$.

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1. Lizorkin kernels. We will use the following notation: $l_{p,q}$, $s_{pq}$, $B^{s}_{pq}$, and $F^{s}_{pq}$ to denote Lorentz sequence space, Schatten class, Besov function space, and Triebel-Lizorkin function space, respectively. By $\pi_p$, $s_n$, and $x_n$ we denote $p$-summing norms, $s$-number function, and Weyl numbers, respectively, see [2, 4, 5].

**Theorem 1.1** (see [1]). Let $s \in \mathbb{R}$, $p \in (0, \infty)$ and let $q, u, v \in (0, \infty]$. Then $B^s_{pu} \subset F^s_{pq} \subset B^s_{pv}$ if and only if

$$0 < u \leq \min(p, q), \quad \max(p, q) \leq v \leq \infty.$$  \hspace{1cm} (1.1)

That is, if and only if $0 < u \leq q \leq v$.

**Proposition 1.2** (see [4]). Let $\Phi \in [B^s_{pu}(0,1), X]$ and $r = \max(p, u)$. Then

$$\Phi_{op} : x \rightarrow (\Phi(\cdot), x)$$  \hspace{1cm} (1.2)

(where $\Phi_{op}$ is an approximable operator from $X'$ into $B^s_{pu}(0,1)$) defines an absolutely $r$-summing operator from $X'$ into $B^s_{pu}(0,1)$. Moreover,

$$\|\Phi_{op} \ | \pi_r \| \leq \|\Phi \ | [B^s_{pu}, X]\|.$$  \hspace{1cm} (1.3)

We restate the previous proposition in the following form in the case of Triebel-Lizorkin space $F^\sigma_{pu}(\Omega)$.

**Proposition 1.3.** Let $X$ be a Banach space, $\Omega \subset \mathbb{R}^N$ a bounded domain, $\sigma > 0$, and $1 \leq p < \infty$. Let $\Phi \in F^\sigma_{pu}(\Omega; X)$ and $r = \max(p, u)$. Then

$$\Phi_{op} : x \rightarrow (\Phi(\cdot), x)$$  \hspace{1cm} (1.4)
defines an absolutely $r$-summing operator from $X'$ into $F_{p,u}^\sigma(X)$. Moreover,

$$\pi_r(\Phi_{\text{op}}) \leq \|\Phi\|_{p,u,\sigma;\Omega,X}.$$  \hfill (1.5)

**Proof.** Given $x_1, \ldots, x_n \in X'$, Jessen's inequality \cite{4} yields

$$\left( \int_\Omega \left[ \sum_{i=1}^n |(\Phi(\xi), x_i)|^r \right]^{p/r} d\xi \right)^{1/p} \leq \left( \sum_{i=1}^n \left[ \int_\Omega |(\Phi(\xi), x_i)|^p d\xi \right]^{r/p} \right)^{1/r}. \hfill (1.6)$$

Therefore,

$$\left\| \left( \sum_{i=1}^n |(\Phi(\cdot), x_i)|^r \right)^{1/r} \right\|_{L_p} \leq \left( \sum_{i=1}^n \| (\Phi(\cdot), x_i) \|_{L_p}^r \right)^{1/r} \leq \|\Phi\|_{L_p} \| (x_i) \|_{\pi_r}. \hfill (1.7)$$

Applying this result to $\Delta_T^m \Phi$, we obtain

$$\left\| \left( \sum_{i=1}^n |(\Delta_T^m \Phi(\cdot), x_i)|^r \right)^{1/r} \right\|_{L_p} \leq \|\Delta_T^m \Phi\|_{L_p} \| (x_i) \|_{\pi_r}. \hfill (1.8)$$

Hence,

$$\left\| \left( \sum_{i=1}^n \left[ \int_\Omega \left[ T^{-\sigma} \left\| (\Delta_T^m \Phi(\cdot), x_i) \right\|_{L_p} \right]^r \frac{d\tau}{\tau} \right]^{u/r} \frac{d\tau}{\tau} \right\|_{L_p}^{1/u} \leq \left( \sum_{i=1}^n \left( \int_\Omega T^{-\sigma} \left\| (\Delta_T^m \Phi(\cdot), x_i) \right\|_{L_p}^u \frac{d\tau}{\tau} \right)^{r/u} \right)^{1/r} \hfill (1.9)$$

Finally, we conclude from the preceding inequalities that

$$\left\| \left( \sum_{i=1}^n |(\Phi(\cdot), x_i)|^r \right)^{1/r} \right\|_{F_{p,u}^\sigma} \leq \left\| \left( \sum_{i=1}^n |(\Phi(\cdot), x_i)|^r \right)^{1/r} \right\|_{L_p} + \left\| \left( \int_\Omega T^{-\sigma} \left| \Delta_T^m \Phi \right|^u \frac{d\tau}{\tau} \right)^{1/u} \right\|_{L_p} \leq \|\Phi\|_{L_p} + \| \Delta_T^m \Phi \|_{L_p} \| (x_i) \|_{\pi_r}. \hfill (1.10)$$

This shows that $\Phi_{\text{op}}$ is absolutely $r$-summing. \hfill \Box
Corollary 1.4 (see [2]). Let \( X \) and \( Y \) be Banach spaces, \( 2 \leq p < \infty \), and \( T \in \pi_{p,2}(X,Y) \). Then \( T \in S^{x}_{p,\infty}(X,Y) \), and for any \( n \in \mathbb{N} \),
\[
x_{n}(T) \leq n^{-1/p} \pi_{p,2}(T).
\]
We are interested in the following theorem.

**Theorem 1.5** (see [3]). Let \( 1 \leq p \leq \max(2,q) \leq \infty \). Then
\[
x_{n}(I_{p,q}^{m}: l_{p}^{m} \rightarrow l_{q}^{m}) \asymp \begin{cases} 
n^{1/q-1/p} & \text{for } 1 \leq p \leq q \leq 2, \\
n^{1/2-1/p} & \text{for } 1 \leq p \leq 2 < q < \infty, \\
1 & \text{for } 2 \leq p \leq q. 
\end{cases}
\]

**Theorem 1.6** (multiplication theorem) [4]). If \( 1/p + 1/q = 1/r \) and \( 1/u + 1/v = 1/w \), then
\[
S(x)^{pu} \circ S(x)^{qv} \subseteq S(x)^{rw}.
\]

**Theorem 1.7** (eigenvalue theorem) [2]). Let \( 0 < p < \infty \), \( 0 < q \leq \infty \), and let \( X \) be a Banach space. Then any operator \( T \in L(X) \) which has Weyl numbers \( (x_{n}(T)) \in l_{p,q} \), \( T \in S^{x}_{p,q}(X) \) is a Riesz operator, the eigenvalue sequence of which is in \( l_{p,q} \), and the following inequality holds
\[
\|(\lambda_{n}(T))\|_{p,q} \leq c \|(x_{n}(T))\|_{p,q}.
\]

2. Eigenvalue theorem for Lizorkin kernels. The following theorem contains the main result of this note.

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^{N} \) be a bounded domain, \( 1 \leq p, q, u, v < \infty \), and \( \sigma + \tau > N(1/p + 1/q - 1) \). Define \( r \) by \( 1/r = (\sigma + \tau)/N + 1/q^{+} \), where \( q^{+} = \max(q',2) \). Then the eigenvalues of any kernel \( k \in F_{p,u}^{\sigma}(\Omega; F_{q,v}^{\tau}(\Omega)) \) belong to the Lorentz sequence space \( l_{r,p} \) with
\[
\|(\lambda_{n}(k))_{n \in \mathbb{N}}\|_{l_{r,p}} \leq c \|k\|_{F_{p,u}^{\sigma}(F_{q,v}^{\tau})}.
\]

The constant \( c \) depends only on the indices and \( \Omega \).

**Proof.** First, we assume that \( p \leq q^{+} \).

We will show that there exists an imbedding map \( \text{id} : F_{p,u}^{\sigma}(\Omega) \rightarrow F_{q,v}^{\tau}(\Omega)^{'} \) and then estimate its Weyl numbers \( x_{n}(\text{id}) \). We factorize an imbedding map \( \text{id} : F_{p,u}^{\sigma}(\Omega) \rightarrow F_{q,v}^{\tau}(\Omega)^{'} \) with the help of maps \( A \) and \( B \) such that
\[
\text{id} = B \circ \text{id}^l \circ A,
\]
\[
F_{p,u}^{\sigma}(\Omega) \xrightarrow{\text{id}} F_{q,v}^{\tau}(\Omega)^{'} \xrightarrow{B} \xrightarrow{A} l_{p}^{m}(\Omega) \rightarrow l_{q}^{m}(\Omega).
\]

(2.2)
This means that

\[ x_n(\text{id}) \leq \|A\| x_n(\text{id}^I) \|B\| \quad (2.3) \]

if we are able to estimate \(\|A\|\) and \(\|B\|\) suitably; from [6], operators \(A\) and \(B\) are defined exactly as they are in [4], and assume that \(\Omega\) contains the unit cube in \(\mathbb{R}^N\) and divide the unit cube in the usual way into \(2^{jN}\) congruent cubes with side length \(2^{-j}\).

From [1], we have

\[ \|A\| \leq c_1 2^{-j(\sigma-N/p)}, \quad \|B\| \leq c_2 2^{j(-\tau-N/q')} \quad (2.4) \]

Substituting (2.4) in (2.3), we get

\[ x_n(\text{id}) \leq c_2 2^{-j(\sigma+\tau)+jN(1/p-1/q')} x_n(\text{id}^I). \quad (2.5) \]

By Theorem 1.5, we have

\[ x_n(\text{id} : F_{pu}^{\sigma}(\Omega) \simeq F_{qv}^{\tau}(\Omega')) < n^{-\rho}, \quad (2.6) \]

where

\[ \rho = \frac{\sigma + \tau}{N} + \begin{cases} 0, & \text{if } 1 \leq p \leq q' \leq 2, \\ \frac{1}{2} - \frac{1}{q}, & \text{if } 1 \leq p \leq 2 \leq q' < \infty, \\ 1 - \frac{1}{p} - \frac{1}{q'}, & \text{if } 2 \leq p \leq q' < \infty, \end{cases} \quad (2.7) \]

and \(n = 2^{Nj}\).

Hence,

\[ \text{id} \in S_{1/\rho, \infty}^{(x)} (F_{pu}^{\sigma}(\Omega) \simeq F_{qv}^{\tau}(\Omega')). \quad (2.8) \]

To estimate the Weyl number of \(T_k\) in \(F_{qv}^{\tau}(\Omega')\), we use the factorization

\[ F_{qv}^{\tau}(\Omega') \overset{T_k}{\rightarrow} F_{pu}^{\sigma}(\Omega) \overset{\text{id}}{\rightarrow} F_{qv}^{\tau}(\Omega'). \quad (2.9) \]

By Proposition 1.3, \(k \in F_{pu}^{\sigma}(\Omega; X)\) implies that \(T_k : X' \simeq F_{pu}^{\sigma}(\Omega)\) is \(p\)-summing. By Corollary 1.4, we have

\[ x_n(T_k : F_{qv}^{\tau}(\Omega') \rightarrow F_{pu}^{\sigma}(\Omega)) \leq \pi_p(T_k) n^{-1/\max(p,2)} \leq c_1 \| k \|_{F_{pu}^{\sigma}(X)} n^{-1/\max(p,2)}, \quad (2.10) \]

that is,

\[ T_k \in S_{s, \infty}^{(x)} (F_{pu}^{\sigma}(\Omega), F_{qv}^{\tau}(\Omega)'), \quad s = \max(p,2). \quad (2.11) \]

We conclude from the multiplication theorem that \(\text{id} \circ T_k \in S_{r,\infty}^{(x)} (F_{qv}^{\tau}(\Omega)'),\) where \(1/r = \rho + 1/s\).
In the case when \( p > q' \), then we have

\[
k \in F^\sigma_{pu} (\Omega; F^T_{qv} (\Omega)) \implies k \in F^\sigma_{q'u} (\Omega; F^T_{qv} (\Omega)). \tag{2.12}
\]

In this way the second case is reduced to the first one.

So, we have shown that the map \( k \to \text{id} \circ T_k \), which assigns to every kernel the corresponding operator, acts as follows:

\[
\text{op} : F^\sigma_{pu} (F^T_{qv}) \twoheadrightarrow S^{(x)}_{r,\infty} (F^T_{qv} (\Omega')). \tag{2.13}
\]

This result can be improved by interpolation. To this end, choose \( p_0, p_1 \), and \( \theta \) such that \( 1/p = 1 - \theta/p_0 + \theta/p_1 \). We now apply the formula

\[
(F^\sigma_{p_0 u} (E), F^\sigma_{p_1 u} (E))_{\theta,p} = F^\sigma_{pu} (E), \quad E = F^T_{qv}. \tag{2.14}
\]

Then, using interpolation as in [2], where \( 1/r = 1 - \theta/r_0 + \theta/r_1 \), hence

\[
(S^{(x)}_{r_0,\infty}, S^{(x)}_{r_1,\infty})_{\theta,p} \subseteq S^{(x)}_{rp}. \tag{2.15}
\]

Hence the interpolation property yields

\[
\text{op} : F^\sigma_{pu} (F^T_{qv}) \twoheadrightarrow S^{(x)}_{r,p} (F^T_{qv} (\Omega')). \tag{2.16}
\]

By the eigenvalue theorem (Theorem 1.7), we therefore obtain \( (\lambda_n(k)) \in l_{r,p} \). □

**Theorem 2.2** (eigenvalue theorem for Sobolev kernels). Let \( 1 \leq p < \infty, 1 < q < \infty, 1/r = m + n + 1/q^+, \) and \( w = \min(q,2) \).

Then

\[
k \in [W^m_p (0,1), W^n_q (0,1)] \implies (\lambda_n(k)) \in l_{r,w}. \tag{2.17}
\]

**Proof.** See [4]. □

The following example proves that our result improves the previous theorem of [4].

**Theorem 2.3.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain, \( 1 \leq p, q, v < \infty, \) and \( \tau > 0 \) with \( \tau > N (1/p + 1/q - 1), p \leq v, \) and \( 1/r := \tau/N + 1/\max(2,q') \). Then the eigenvalues of any kernel \( k \in L_p (F^T_{qv}) \) belong to the Lorentz sequence space \( l_{r,v} \) with

\[
\|(\lambda_n(k))_{n \in \mathbb{N}}\|_{l_{r,v}} \leq c \|k\|_{L_p (F^T_{qv})}. \tag{2.18}
\]

**Proof.** We may assume that \( p \leq q' \). Then, reasoning similarly as in the proof of Theorem 2.1, it follows that the map \( k \to T_k \), which assigns to every kernel the corresponding operator, acts as follows:

\[
\text{op} : L_p (F^T_{qv}) \twoheadrightarrow S^{(x)}_{r,\infty} (L_p (\Omega)). \tag{2.19}
\]
This result can be improved by interpolation. To this end, we apply the imbedding
\[ (L_p, (E_0, E_1))_{\vartheta, m} \subseteq ((L_p, E_0), (L_p, E_1))_{\vartheta, m}, \quad p < m, \]  
(2.20)
to the interpolation couple \((F_{q_0}, F_{q_1})\). The interpolation property now implies that
\[ \text{op}: L_p(F_{q_0}^\tau) \rightarrow S_{r, \psi}^{(\infty)}(L_p(\Omega)). \]  
(2.21)
By the eigenvalue theorem (Theorem 1.7), we therefore obtain \((\lambda_n(k)) \in l_{r, \psi}\). □

**Example 2.4.** (1) In this example, we will indicate a special case of the Lizorkin space \(F_{p_q}^\sigma(\mathbb{R}^N)\). When \(1 < p < \infty\) and \(s \in \mathbb{N}_0\), then
\[ F_{p_q}^s(\mathbb{R}^N) = W_p^s(\mathbb{R}^N) \]  
(2.22)
are the classical Sobolev spaces.
We compare this case with Theorem 2.2. We find that
\[ k \in W_p^\sigma(W_0^\tau) \Rightarrow (\lambda_n(k)) \in l_{r, \psi}, \]  
(2.23)
where \(w = \min(q, 2)\), and
\[ k \in F_{p_q}^\sigma(F_{q_0}^\tau) \Rightarrow (\lambda_n(k)) \in l_{r, \psi}. \]  
(2.24)
We conclude that if \(p < w = \min(q, 2)\), \(2 \leq q < \infty\), \(1 < p < 2\), then
\[ l_{r, \psi} \subset l_{r, \psi}, \]  
(2.25)
that is,
\[ \|(\lambda_n(k))_{n \in \mathbb{N}}\|_{r, \psi} \leq \|(\lambda_n(k))_{n \in \mathbb{N}}\|_{r, \psi}. \]  
(2.26)
(2) We compare
\[ k \in W_p^\sigma(W_0^\tau) \Rightarrow (\lambda_n(k)) \in l_{r, \psi}, \]  
(2.27)
where \(w = \min(q, 2)\), with
\[ k \in L_p(F_{q}^\tau) \Rightarrow (\lambda_n(k)) \in l_{r, \psi}, \]  
(2.28)
where \(p \leq v\). We conclude that if \(v < w = \min(q, 2)\), \(2 \leq q < \infty\), \(1 < p < 2\), then
\[ l_{r, \psi} \subset l_{r, \psi}, \]  
(2.29)
that is,
\[ \|(\lambda_n(k))_{n \in \mathbb{N}}\|_{r, \psi} \leq \|(\lambda_n(k))_{n \in \mathbb{N}}\|_{r, \psi}. \]  
(2.30)
References


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