POWERSUM FORMULA FOR DIFFERENTIAL RESOLVENTS

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We will prove that we can specialize the indeterminate $\alpha$ in a linear differential $\alpha$-resolvent of a univariate polynomial over a differential field of characteristic zero to an integer $q$ to obtain a $q$-resolvent. We use this idea to obtain a formula, known as the powersum formula, for the terms of the $\alpha$-resolvent. Finally, we use the powersum formula to rediscover Cockle’s differential resolvent of a cubic trinomial.

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1. Introduction. It was proved in [4, Theorem 37, page 67] that for any integer $q$, a polynomial $P(t) \equiv \sum_{k=0}^{N} (-1)^{N-k} e_{N-k} t^k$ of a single variable $t$ whose coefficients $\{e_{N-k}\}_{k=0}^{N}$ lie in an ordinary differential ring $\mathbb{R}$ with derivation $D$ possesses an ordinary linear differential $\alpha$-resolvent and an ordinary linear differential $q$-resolvent, where $\alpha$ is a constant, transcendental over $\mathbb{R}$. With no loss of generality, we assume that $P$ is monic and has no zero roots. Then the coefficient $e_{N-k}$ is the $(N-k)$th elementary symmetric function of the roots of $P$. We assume that $P(t) = \prod_{k=1}^{n} (t - z_k)^{\pi_k}$ has $n \leq N$ distinct roots and $\pi_k$ is the multiplicity of the root $z_k$ in $P$. It was proved in [2] that for each root $z_k$, there exists a nonzero solution $y_k$ of the logarithmic differential equation $Dy_k / y_k = \alpha \cdot (Dz_k / z_k)$. Obviously, such solutions are unique only up to a constant multiple. We define the notation $z_k^\alpha$ to represent any such solution $y_k$. Hence, we will call $y_k$ an $\alpha$-power of $z_k$. From now on, we will drop the subscript $k$ on $z_k$ and $y_k$. It will be understood that a different $z$ implies a different $y$.

In this paper, we present the powersum formula as a new method for computing resolvents, although it remains a conjecture whether the powersum formula always yields a (nonzero) resolvent rather than an identically zero equation. It was proved in [5, Theorem 4.1, page 726] that if all the distinct roots of a polynomial are differentially independent over constants, then the powersum formula yields a resolvent. It was shown in [6, Section 11, pages 344-345], how the solution of the Riccati nonlinear differential equation is related to the resolvent of a quadratic polynomial.

2. Notation. Let $\mathbb{N}$ denote the set of positive integers. Let $\mathbb{N}_0$ denote the set of nonnegative integers. Let $\mathbb{Z}^\#$ denote the set of nonzero integers. The following notation has been slightly modified from Kolchin’s notation in [2] and Macdonald’s notation in [3]. Let $\mathbb{Z}\{e\}$ denote the differential ring generated by the integers $\mathbb{Z}$ and the $N$ coefficients $e \equiv \{e_k\}_{k=1}^{N}$ of $t$ in $P$. Let $\mathbb{Q}\{e\}$ denote the differential field generated by the rational numbers $\mathbb{Q}$ and $e$. For each $m \in \mathbb{N}$, let $\mathbb{Z}\{e\}_m$ denote the ordinary (nondifferential) ring
generated by \(Z, e\), and the first \(m\) derivatives of \(e\). (A differential ring must contain infinitely many derivatives of any of its elements.) Let \(\mathbb{Q}(e)_m(z) = \mathbb{Q}(e)_m[z]\) denote the field generated by \(\mathbb{Q}, e\), the first \(m\) derivatives of \(e\), and the single root \(z\). From this point on, we will write \(\mathbb{Q}(e)_m[z]\) instead of \(\mathbb{Q}(e)_m(z)\) for this field to emphasize the fact that elements in this field are polynomial in the root \(z\). If \(\mathbb{R}\) represents any of the rings or fields mentioned so far, then let \(\mathbb{R}[t, \alpha]\) denote the polynomial ring in the indeterminates \(t\) and \(\alpha\) over \(\mathbb{R}\).

Let \(\theta \equiv (n!) \cdot (\prod_{k=1}^{n} \pi_k \cdot (\prod_{k=1}^{n} z_k) \cdot (\prod_{i<j}(z_i - z_j)^2)\). By our conditions on \(P\), \(\theta \neq 0\). It is also easy to show that \(\theta \in \mathbb{Z}[e]\). For each \(m \in \mathbb{N}\), it was proved in [4, Theorem 32, page 60] that there exists a polynomial \(G_m(t, \alpha)\) in \(t\) and \(\alpha\) satisfying the following definition.

**Definition 2.1.** Define \(G_m(t, \alpha)\) to be the polynomial in \(t\) and \(\alpha\) such that \(G_m(z, \alpha) = D^m y / (\alpha \cdot y)\) for each root \(z\) of \(P\) and \(D^m \cdot G_m(t, \alpha) \in \mathbb{Z}[e]_m[t, \alpha]\).

A specialization \(\phi\) is a ring homomorphism \(\phi : \mathbb{R} \rightarrow \mathbb{R}\) from a ring \(\mathbb{R}\) into an integral domain \(\hat{\mathbb{R}}\). For any polynomial \(P(t) = \sum_{N=0}^{N} (-1)^{N-k} e_{N-k} \cdot t^k \in \mathbb{R}[t]\), \(\phi(P)\) is defined to be the polynomial \((\phi P)(t) = \sum_{N=0}^{N} (-1)^{N-k} \phi(e_{N-k}) \cdot t^k \in \mathbb{R}[t]\). A differential specialization \(\phi\) is a specialization \(\phi : \mathbb{R} \rightarrow \mathbb{R}\) from a differential ring \(\mathbb{R}\) with derivation \(D\) into a differential integral domain \(\hat{\mathbb{R}}\) with derivation \(\hat{D}\) such that \(\phi D = \hat{D} \phi\) on \(\mathbb{R}\).

3. **Specializing \(\alpha\).** Let \(q \in \mathbb{N}\). Let \(\phi_q : \mathbb{Q}(e)[z, \alpha] \rightarrow \mathbb{Q}(e)[z]\) be the ring specialization such that \(\phi_q\) is the identity on \(\mathbb{Q}(e)[z]\) and \(\phi_q(\alpha) = q\). We may compute \(Dz^q / (q \cdot z^q)\). Since \(\phi_q\) is not defined to act on \(y\), we are not able to specialize \(y\) to \(z^q\) in Theorem 3.1. However, \(\phi_q\) is defined to act on \(D^m y / (\alpha \cdot y)\) since \(D^m y / (\alpha \cdot y) = G_m(z, \alpha)\). Let \(\phi_q(z) = D^m z^q / (q \cdot z^q)\). Theorem 3.1 asserts that \(G_m(z, q) = D^m z^q / (q \cdot z^q)\). Theorem 3.2 asserts that \(\phi_q(D^m y / (\alpha \cdot y)) = D^m z^q / (q \cdot z^q)\).

**Theorem 3.1.** Assume all the same definitions and notations as in the introduction. Then the \(m\)th derivative of \(z^q\) can be expressed as a product of \(q \cdot z^q\) and an element in \(\mathbb{Q}(e)_m[z]\). More specifically, \(G_m(z, q) = D^m z^q / (q \cdot z^q)\), where \(G_m(z, q) \in \mathbb{Q}(e)_m[z]\) and \(G_m(t, \alpha)\) was given in Definition 2.1.

**Proof.** For brevity, write \(G_m = G_m(z, \alpha)\) for the particular root \(z\). We emphasize that \(G_m\) is \(G_m(t, \alpha)\) with \(t\) specialized to the particular root \(z\). We find that \(\theta \cdot (Dz^q / (q \cdot z^q)) = \theta \cdot (Dz / z) = \theta \cdot G_1 \in \mathbb{Z}[e]_1[z]\). Therefore,

\[
Dz^q = q \cdot z^q \cdot G_1 \Rightarrow D^2 z^q = q \cdot (q \cdot z^{q-1} Dz \cdot G_1 + z^q \cdot DG_1) = q \cdot z^q \left( q \cdot \left( \frac{Dz}{z} \right) \cdot G_1 \right) + q \cdot z^q \cdot G_2 = q \cdot z^q \cdot \phi_q(G_2).
\]

So, \(D^m z^q = q \cdot z^q \cdot \phi_q(G_m)\) is true for \(m = 1\). Now assume that it is true for \(m \geq 2\). Then \(D^{m+1} z^q = q \cdot (q \cdot z^{q-1} (Dz) \cdot \phi_q(G_m) + z^q \cdot D(\phi_q(G_m)))\). But \(\phi_q\) specializes \(\alpha\), whose derivative is 0, to an integer whose derivative is 0. Thus, \(D(\phi_q(G_m)) = \phi_q(D(G_m))\).
Hence,

\[
D^{m+1}z^q = q \cdot (q \cdot z^{q-1} \cdot (Dz) \cdot \phi_q(G_m) + z^q \cdot \phi_q(D(G_m))) \\
= q \cdot z^q \cdot \left( q \cdot \frac{Dz}{z} \cdot \phi_q(G_m) + \phi_q(D(G_m)) \right) \\
= q \cdot z^q \cdot (\phi_q(\alpha) \cdot G_1 + \phi_q(G_m) + \phi_q(D(G_m))) \\
= q \cdot z^q \cdot \phi_q(G_{m+1}).
\]

Therefore, \( D^{m+1}z^q = q \cdot z^q \cdot G_{m+1}(z,q) \) since \( \phi_q \) affects only \( \alpha \). By the principle of mathematical induction, this equation is true for all positive integers \( m \).

Just because \( Dy/(\alpha \cdot y) = Dz/z = Dz^q/(q \cdot z^q) \) implies that \( Dy/(\alpha \cdot y) \) is independent of \( \alpha \), it does not follow that \( D^m y/(\alpha \cdot y) \) is independent of \( \alpha \) for \( m \geq 2 \). We can see this by observing that \( D^m y/(\alpha \cdot y) \neq D^m z/z \neq D^m z^q/(q \cdot z^q) \neq D^m y/(\alpha \cdot y) \) for \( m \geq 2 \).

**Theorem 3.2.** Assume all the same definitions and notations as in Theorem 3.1 and Section 2. Then, for each \( m \in \mathbb{N} \), the specialization under \( \phi_q \) of \( D^m y/(\alpha \cdot y) \) is \( D^m z^q/(q \cdot z^q) \). That is, \( \phi_q(D^m y/(\alpha \cdot y)) = D^m z^q/(q \cdot z^q) \).

**Proof.** By Definition 2.1, \( D^m y/(\alpha \cdot y) = G_m(z,\alpha) \). By Theorem 3.1, \( D^m z^q/(q \cdot z^q) = G_m(z,q) \). Putting these results together yields

\[
\phi_q\left( \frac{D^m y}{\alpha \cdot y} \right) = \phi_q(G_m(z,\alpha)) = G_m(z,q) = \frac{D^m z^q}{q \cdot z^q}.
\]

**4. Powersum satisfaction theorem and formula.** An \( \alpha \)-resolvent of a polynomial \( P(t) = \sum_{i=0}^{N} (-1)^{n-i}e_{N-i}t^i \in \mathbb{F}[t] \) over a differential field \( \mathbb{F} \) with derivation \( D \) is a linear ordinary differential equation \( \sum_{m=0}^{\alpha} B_m(\alpha) \cdot D^m y = 0 \) of finite order \( \alpha \) such that each of the coefficient functions \( B_m(\alpha) \) lies in the field \( \mathbb{Q}(e)/(\alpha) \) (or preferably in the ring \( \mathbb{Z}[e][\alpha] \)) such that not all \( B_m(\alpha) \) are identically zero, and which is satisfied by the \( \alpha \)-power of every root \( z \) of \( P \). In other words, the coefficient functions of the resolvent are independent of the choice of root and are not all zero. By [4, Theorem 37, page 67], resolvents for any polynomial are guaranteed to exist. We state this assertion in Theorem 4.1.

**Theorem 4.1.** Let \( P(t) = \sum_{i=0}^{N} (-1)^{n-i}e_{N-i}t^i \in \mathbb{F}[t] \) be a polynomial of degree \( N \) in \( t \) over a \( d \)-field \( \mathbb{F} \) with \( n \) distinct roots \( \{z_i\}_{i=1}^{n} \). Then there exists an \( \alpha \)th order differential resolvent \( \sum_{m=0}^{\alpha} B_m(\alpha) \cdot D^m y = 0 \) with \( B_m(\alpha) \in \mathbb{Z}[e][\alpha] \), \( B_0(0) = 0 \), and \( \deg_{\alpha} B_m(\alpha) \leq o(\alpha - 1)/2 - m + 1 \) for some \( o \in [\alpha] \). Furthermore, \( o \) may be chosen to equal the number of \( \{y_j\}_{j=1}^{\alpha} \) linearly independent over constants, and all solutions of this resolvent are linear combinations over constants of these \( o \) \( y_j \)'s.

Theorem 4.1 gives us an upper bound on the degree in \( \alpha \) in an \( \alpha \)-resolvent of \( P \). Theorem 4.2 allows us to specialize the indeterminate \( \alpha \) to an integer \( q \) (or any number) to obtain a \( q \)-resolvent.
Theorem 4.2 (powersum satisfaction theorem). Let $P \in \mathbb{F}[t]$ be a monic polynomial with $n$ distinct roots $z = \{z_i\}_{i=1}^n$, none of which is zero and not all of which are constants. Let $q \in \mathbb{Z}$. If $R_\alpha \equiv \sum_{m=0}^\infty B_m(\alpha) \cdot D^m y$ is an $\alpha$-resolvent for $P$ of arbitrary order $o$, where $B_m(\alpha) = \sum_{i \geq 0} b_{i,m} \alpha^i \in \mathbb{Z}[\epsilon][\alpha]$, with $b_{i,m} \in \mathbb{Z}[\epsilon]$, then $R_\alpha$ specializes to the $q$-resolvent $R_q \equiv \sum_{m=0}^\infty B_m(q) \cdot D^m y$ for $q \in \mathbb{Z}^\#$ under $\phi_q(\alpha) = q$ and $\phi_q(u) = u$ for each $u \in \mathbb{Z}[\epsilon]$. Furthermore, the $q$th powersum formula $p_q$ satisfies $\sum_{m=0}^\infty B_m(q) \cdot D^m p_q = 0$ for each $q \in \mathbb{Z}^\#$. 

Proof. By Definition 2.1 of $G_m(t, \alpha)$, we have

$$\sum_{m=0}^o B_m(\alpha) \cdot D^m y = 0 \iff \sum_{m=0}^o B_m(\alpha) \cdot \frac{D^m y}{\alpha^q} = 0 \iff \sum_{m=0}^o B_m(\alpha) \cdot G_m(z, \alpha) = 0. \tag{4.1}$$

Now for each $q \in \mathbb{Z}^\#$, specialize this equation under $\phi_q$ to get $\sum_{m=0}^o B_m(q) \cdot G_m(z, q) = 0$ by Theorem 3.2, since $\phi_q | \mathbb{F} = I$. For any of the roots of $P$, we have $G_m(z, q) = D^m z^q / (q \cdot z^q)$ by Theorem 3.1. Thus,

$$\sum_{m=0}^o B_m(q) \cdot \frac{D^m z^q}{q \cdot z^q} = 0 \iff \sum_{m=0}^o B_m(q) \cdot D^m z^q = 0 \tag{4.2}$$

for each $q \in \mathbb{Z}^\#$. Therefore, an $\alpha$-resolvent specializes to a $q$-resolvent for each $q \in \mathbb{Z}^\#$ under $\phi_q$. Now sum over the $N$ roots of $P$ including their multiplicities to get $\sum_{m=0}^o B_m(q) \cdot D^m p_q = 0$ for each $q \in \mathbb{Z}^\#$. $\Box$

The powersum satisfaction theorem states that for any monic polynomial $P$, the coefficients $b_{i,m}$ of $\alpha$ in any $\alpha$-resolvent $R_\alpha \equiv \sum_{(i,m) \in S} b_{i,m} \cdot \alpha^i D^m y$ of $P$ satisfy an infinite system of homogeneous equations

$$[q^i D^m p_q]_{q \leq (i,m) \leq \infty \ (i,m) \in S} \cdot [b_{i,m}]_{(i,m) \in S} = [0_q]_{1 \leq q < \infty}. \tag{4.3}$$

Here, $S$ denotes the set of pairs $(i, m)$ consisting of a power of $\alpha$, denoted by $i$, and an order of a derivative, denoted by $m$, such that $b_{i,m} \neq 0$. Let $|S|$ denote the size of $S$. We will be interested in proving that the rank

$$\text{rk} [q^i D^m p_q]_{q \leq (i,m) \leq \infty \ (i,m) \in S} \tag{4.4}$$

of the matrix $[q^i D^m p_q]_{q \leq (i,m) \leq \infty \ (i,m) \in S}$ equals $|S| - 1$ under certain circumstances. Under those circumstances, one can solve this system of equations to get a nonzero solution for $b_{i,m}$. The solution is given by $b_{i,m} = F_{i,m} = (-1)^{\text{sgn}(i,m)} \cdot q^{\Gamma'} D^{\Gamma'} p_q |_{(i',m')} \ (i',m') \in (i,m) \in S$, where $\text{sgn}(i,m)$ indicates the ordering of the term $b_{i,m}$ in the resolvent, and we take $\Gamma'$ to be the smallest possible set of positive integers that will guarantee a nonzero solution. In numerous examples, it has been found that $\Gamma \equiv \{k \in \mathbb{N} \mid 1 \leq k \leq |S| - 1\}$. We call this the powersum formula for a resolvent of $P$. We use the notation $F_{i,m}$ to denote the terms of the resolvent obtained by this method to suggest the word formula. We will denote the resolvent obtained by this formula by $\mathcal{R}_\alpha$. So, $\mathcal{R}_\alpha = \{F_{i,m}\}$. 


If \( \text{rk}[q^i D^m p_q]_{q \times (i,m) \mid 1 \leq q < \infty, (i,m) \in S} = |S| \), then the only solution would be \( b_{i,m} = 0 \) for all \((i,m) \in S\), contradicting the hypothesis that \( R_\alpha \) is nonzero. Unfortunately, for a given polynomial \( P \), one does not know a priori what the set \( S \) of nonzero \( b_{i,m} \) is or how large it is. Nevertheless, we may summarize the results obtained so far in a corollary to the powersum satisfaction theorem.

**Corollary 4.3** (the powersum formula). Let \( R_\alpha = \sum_{(i,m) \in S} b_{i,m} \cdot \alpha^i D^m y \) be an \( \alpha \)-resolvent of \( P \), where \( S \subset \mathbb{N}_0 \times \mathbb{N}_0 \) is a finite set. If there exists a set of \(|S| - 1\) integers \( \Gamma \subset \mathbb{N} \) such that not all the \( F_{i,m} \) given by the powersum formula \( F_{i,m} \equiv (-1)^{\text{sgn}(i,m)} \cdot |q^i D^m p_q|_{(i',m') \in (i,m) \in \Gamma} \) are zero, then the linear ordinary differential equation (ODE), \( R_\alpha = \sum_{(i,m) \in S} F_{i,m} \cdot \alpha^i D^m y \), is an integral \( \alpha \)-resolvent of \( P \). If no such set of integers \( \Gamma \) exists, then the powersum formula yields all zeroes for \( F_{i,m} \).

The author believes that the resolvent \( R_\alpha \) given by the powersum formula will be a \( \mathbb{Q}(e) \)-multiple, not just a \( \mathbb{Q}(e)(\alpha) \)-multiple of \( R_\alpha \), but this requires proof. For example, let \( \alpha^m \cdot D^H y \) denote the highest power of \( \alpha \) on the highest derivative of \( y \) in \( R_\alpha \). Even though \( F_{M,L}/b_{M,L} \cdot R_\alpha \) and \( R_\alpha \) are both resolvents (provided that \( F_{M,L} \neq 0 \)) with the same coefficient function of \( \alpha^m \cdot D^H y \), one must eliminate the possibility that their other terms may differ due to the possibility that \( P \) has resolvents of lower order.

**5. Example.** We will now apply the powersum formula to compute a particular \( \alpha \)-resolvent of a particular trinomial. It has not yet been proved that this formula yields a nonzero differential equation for every polynomial. However, in every polynomial the author has tested, it has been possible to set up an \( \alpha \)-resolvent, itself a polynomial in the power \( \alpha \), and choose the proper set of powersums such that the powersum formula yields a nonzero answer. If the powersum formula yields a nonzero answer, then it is guaranteed by Corollary 4.3 that the answer is a (nonzero) resolvent of the polynomial. By a very long and difficult proof in [4, Theorem 41, page 74] and [5, Theorem 4.1, page 726], it has been shown that in case the distinct roots of the polynomial are differentially independent over constants (i.e., they satisfy no polynomial differential equations over \( \mathbb{Q} \)), then the powersum formula yields a nonzero resolvent.

The powersum formula has the advantage of giving a resolvent in an integral form. In the next example, this means the powersum formula gives a resolvent all of whose terms lie in the ring \( \mathbb{Z}[x,\alpha] \).

**Example 5.1** (Sir James Cockle’s resolvent of a trinomial). Cockle [1] gave a formula for a linear differential \( \alpha \)-resolvent (although he did not call it that) for any trinomial of the form \( t^n + x \cdot t^p - 1 \), where \( Dx \equiv 1 \). Consider the particular trinomial \( P(t) \equiv t^3 + x \cdot t^2 - 1 \), where \( n = 3 \) and \( p = 2 \). Then, Cockle’s resolvent specializes to \( 27 \cdot D^3 y = 4 \cdot (x \cdot D + \alpha/2)(x \cdot D + 3/2 + \alpha/2)(x \cdot D - \alpha/2)(x \cdot D - \alpha^2 \cdot (3 + \alpha)) y \). This expands to \( 27 \cdot D^3 y = (4 \cdot (x \cdot D)^3 + 6 \cdot (x \cdot D)^2 - 3 \cdot \alpha \cdot (1 + \alpha) \cdot (x \cdot D) - \alpha^2 \cdot (3 + \alpha)) y \). Replacing \( (x \cdot D)^3 \) with \( x^3 \cdot D^3 + 3 \cdot x^2 \cdot D^2 + x \cdot D \) and \( (x \cdot D)^2 \) with \( x^2 \cdot D^2 + x \cdot D \) yields \( (4x^3 - 27) \cdot D^3 y + 18 \cdot x^2 \cdot D^2 y + (10 - 3 \cdot \alpha - 3 \cdot \alpha^2) \cdot x \cdot D y - \alpha^2 \cdot (3 + \alpha) \cdot y = 0 \), which has the form \( f_1 \cdot D^3 y + f_2 \cdot D^2 y + f_3 \cdot f_4 \cdot \alpha + f_5 \cdot \alpha^2 \cdot D y + (f_6 \cdot \alpha^2 + f_7 \cdot \alpha^3) \cdot y = 0 \). The powersum formula requires one to know a priori the various powers of \( \alpha \) appearing in a resolvent. Specialize \( \alpha \) to one of the six integers \( q \in \{1,2,3,4,5,6\} \), then sum the resulting equation over each of the three
roots. Doing this for each \( q \in \{1, 2, 3, 4, 5, 6\} \), one gets a system of six linear equations in the undetermined coefficient functions \( \{f_k\}_{k=1}^7 \) of the form \( \mathfrak{N} \cdot \vec{f} = \vec{0} \), where \( \mathfrak{N} \) is the 6×7 matrix defined by

\[
\mathfrak{N} = \begin{bmatrix}
D^3 p_1 & D^2 p_1 & D p_1 & 1 \cdot D p_1 & 1^2 \cdot p_1 & 1^3 \cdot p_1 \\
D^3 p_2 & D^2 p_2 & D p_2 & 2 \cdot D p_2 & 2^2 \cdot p_2 & 2^3 \cdot p_2 \\
D^3 p_3 & D^2 p_3 & D p_3 & 3 \cdot D p_3 & 3^2 \cdot p_3 & 3^3 \cdot p_3 \\
D^3 p_4 & D^2 p_4 & D p_4 & 4 \cdot D p_4 & 4^2 \cdot p_4 & 4^3 \cdot p_4 \\
D^3 p_5 & D^2 p_5 & D p_5 & 5 \cdot D p_5 & 5^2 \cdot p_5 & 5^3 \cdot p_5 \\
D^3 p_6 & D^2 p_6 & D p_6 & 6 \cdot D p_6 & 6^2 \cdot p_6 & 6^3 \cdot p_6
\end{bmatrix},
\]

(5.1)

\( \vec{f} \) is the 7×1 column vector defined by

\[
\vec{f} = \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
f_7
\end{bmatrix},
\]

(5.2)

and \( \vec{0} \) is the 6×1 column vector defined by

\[
\vec{0} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

(5.3)

The following program, written in Mathematica 4.0 for Students and run on a Dell Dimension XPS R400 computer using Windows 98 operating system, computes the seven terms \( \{f_k\}_{k=1}^7 \) by setting each \( f_k \) to the appropriate cofactor of \( \mathfrak{N} \). This matrix is denoted by \( T \) in the program. The symbol \( s[k] \) stands for the \( k \)th powersum \( p_k \) of the roots of \( P \). The output is denoted by \( f \), which is defined as the transpose of \( \vec{f} \). The result is

\[
\begin{bmatrix}
-27 + 4x^3 & 18x^2 & 10x & -3x & -3x & -3 & -1
\end{bmatrix},
\]

(5.4)

which is the Cockle resolvent. The computation time is less than 5 seconds.

\[
x=.; \ s[0]=3; \ s[1]=-x; \ s[2]=x^2;
\]

\[
\text{Table}[s[k+3]=Expand[-x^8 s[k+2]+s[k]],\{k,0,3\}];
\]

\[
T=\text{Table}[\{D[s[k],\{x,3\}],D[s[k],\{x,2\}],D[s[k],x],
k^3 D[s[k],x], k^2 D[s[k],x], k D[s[k],x], \},\{k,1,6\}];
\]

\[
M=\text{Minors}[T,6];
\]

\[
f=\text{Table}[\text{Simplify}[m[[1,k]]*(-1)^(7-k)/(466560*x)],\{k,1,7\}].
\]
To see the output in Mathematica for other variables, remove the semicolon after its formula. For the record, the first six powersums are (written in the form Mathematica gives) \( p_1 = -x, p_2 = x^2, p_3 = 3 - x^3, p_4 = -4x + x^4, p_5 = 5x^2 - x^5, \) and \( p_6 = 3 - 6x^3 + x^6. \) The cofactors of the matrix \( M \) had to be divided by 466560 \( \times x = 2^7 \cdot 3^6 \cdot 5^1 \cdot x \) to get the resolvent in Cohnian form, that is, such that the only divisors in \( \mathbb{Z}[x, \alpha] \) among all the terms of the resolvent are \( \pm 1. \)

References


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