ON THE LAGRANGE RESOLVENTS OF A DIHEDRAL QUINTIC POLYNOMIAL

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The cyclic quartic field generated by the fifth powers of the Lagrange resolvents of a dihedral quintic polynomial \( f(x) \) is explicitly determined in terms of a generator for the quadratic subfield of the splitting field of \( f(x) \).

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Let \( f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbb{Q}[x] \) be an irreducible quintic polynomial with a solvable Galois group. Let \( x_1, x_2, x_3, x_4, x_5 \in \mathbb{C} \) be the roots of \( f(x) \). The splitting field of \( f \) is \( K = \mathbb{Q}(x_1, x_2, x_3, x_4, x_5) \). Let \( \zeta \) be a primitive fifth root of unity. The Lagrange resolvents of the root \( x_1 \) are

\[
\begin{align*}
r_1 &= (x_1, \zeta) = x_1 + x_2 \zeta + x_3 \zeta^2 + x_4 \zeta^3 + x_5 \zeta^4 \in K(\zeta), \\
r_2 &= (x_1, \zeta^2) = x_1 + x_2 \zeta^2 + x_3 \zeta^4 + x_4 \zeta + x_5 \zeta^3 \in K(\zeta), \\
r_3 &= (x_1, \zeta^3) = x_1 + x_2 \zeta^3 + x_3 \zeta + x_4 \zeta^4 + x_5 \zeta^2 \in K(\zeta), \\
r_4 &= (x_1, \zeta^4) = x_1 + x_2 \zeta^4 + x_3 \zeta^3 + x_4 \zeta^2 + x_5 \zeta \in K(\zeta).
\end{align*}
\]

(1)

We set

\[ R_i = r_5^i, \quad i = 1, 2, 3, 4. \]

(2)

By [1, Theorem 2] we know that the Galois group of \( f \) is \( \mathbb{Z}_5 \) (cyclic group of order 5), \( D_5 \) (dihedral group of order 10), or \( F_{20} \) (Frobenius group of order 20). When \( \text{Gal}(f) \cong D_5 \), the splitting field \( K \) of \( f \) contains a unique quadratic subfield, say \( \mathbb{Q}(\sqrt{m}) \) (\( m \) square-free integer \( \neq 1 \)). In this note we show, for quintic polynomials \( f \) with \( \text{Gal}(f) \cong D_5 \), that the fields \( \mathbb{Q}(R_i) \) (\( i = 1, 2, 3, 4 \)) are the same cyclic quartic field and we give a simple explicit generator for this field. We prove the following theorem.

**Theorem 1.** If \( \text{Gal}(f) \cong D_5 \), then

\[ \mathbb{Q}(R_i) = \mathbb{Q}\left(\sqrt{-m(5 + 2\sqrt{5})}\right), \quad i = 1, 2, 3, 4, \]

(3)

where \( \mathbb{Q}(\sqrt{m}) \) is the unique quadratic subfield of the splitting field \( K \) of \( f \).
PROOF. Expanding \((x_1, \zeta)^5 = (x_1 + x_2 \zeta + x_3 \zeta^2 + x_4 \zeta^3 + x_5 \zeta^4)^5\) we obtain

\[ R_1 = l_0 + l_1 \zeta + l_2 \zeta^2 + l_3 \zeta^3 + l_4 \zeta^4, \]

where \(l_0, l_1, l_2, l_3, l_4 \in K\) are given in [1, page 391] and satisfy

\[ l_0 + l_1 + l_2 + l_3 + l_4 = (x_1 + x_2 + x_3 + x_4 + x_5)^5 = 0. \]

As \(\text{Gal}(f) \cong D_5\), by [1, Theorem 2, page 397] the discriminant \(D\) of \(f\) is a square in \(\mathbb{Q}\). Thus, by [1, pages 392–397], \(l_1, l_2, l_3, l_4\) are the roots of a quartic polynomial belonging to \(\mathbb{Q}[x]\), which factors over \(\mathbb{Q}\) into two irreducible conjugate quadratics

\[ (x^2 + (T_1 + T_2 \sqrt{D}) x + (T_3 + T_4 \sqrt{D}))(x^2 + (T_1 - T_2 \sqrt{D}) x + (T_3 - T_4 \sqrt{D})) \]

with \(T_1, T_2, T_3, T_4 \in \mathbb{Q}\). The roots of one of these quadratics (without loss of generality the first) are \(l_1\) and \(l_4\), and the roots of the other are \(l_2\) and \(l_3\). Thus

\[ l_1 + l_4 = -T_1 - T_2 \sqrt{D}, \quad l_2 + l_3 = -T_1 + T_2 \sqrt{D}, \]
\[ l_1 l_4 = T_3 + T_4 \sqrt{D}, \quad l_2 l_3 = T_3 - T_4 \sqrt{D}. \]

Clearly \([\mathbb{Q}(l_i) : \mathbb{Q}] = 2\) \((i = 1, 2, 3, 4)\). Also \(l_i \in K\) \((i = 1, 2, 3, 4)\) so that \(\mathbb{Q}(l_i) \subseteq K\) \((i = 1, 2, 3, 4)\). However \(K\) has a unique quadratic subfield \(\mathbb{Q}(\sqrt{m})\). Thus \(\mathbb{Q}(l_i) = \mathbb{Q}(\sqrt{m})\), \(i = 1, 2, 3, 4\). Hence

\[ l_1 = a + b \sqrt{m}, \quad l_4 = a - b \sqrt{m}, \quad l_2 = c + d \sqrt{m}, \quad l_3 = c - d \sqrt{m}, \]

where \(a, b, c, d \in \mathbb{Q}, b \neq 0\) and \(d \neq 0\). Thus

\[ l_0 = -l_1 - l_2 - l_3 - l_4 = -2a - 2c. \]

Next we define

\[ g(x) = (x - R_1)(x - R_2)(x - R_3)(x - R_4) \in K(\zeta)[x]. \]

Hence, as \(1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0\), we obtain

\[ R_1 = l_0 + l_1 \zeta + l_2 \zeta^2 + l_3 \zeta^3 + l_4 \zeta^4 \]
\[ = (a + b \sqrt{m} + 2a + 2c) \zeta + (c + d \sqrt{m} + 2a + 2c) \zeta^2 \]
\[ + (c - d \sqrt{m} + 2a + 2c) \zeta^3 + (a - b \sqrt{m} + 2a + 2c) \zeta^4 \in \mathbb{Q}(\sqrt{m}, \zeta). \]

Similarly

\[ R_2 = (a + b \sqrt{m} + 2a + 2c) \zeta^2 + (c + d \sqrt{m} + 2a + 2c) \zeta^4 \]
\[ + (c - d \sqrt{m} + 2a + 2c) \zeta + (a - b \sqrt{m} + 2a + 2c) \zeta^3 \in \mathbb{Q}(\sqrt{m}, \zeta), \]
\[ R_3 = (a + b \sqrt{m} + 2a + 2c) \zeta^3 + (c + d \sqrt{m} + 2a + 2c) \zeta \]
\[ + (c - d \sqrt{m} + 2a + 2c) \zeta^4 + (a - b \sqrt{m} + 2a + 2c) \zeta^2 \in \mathbb{Q}(\sqrt{m}, \zeta), \]
\[ R_4 = (a + b \sqrt{m} + 2a + 2c) \zeta^4 + (c + d \sqrt{m} + 2a + 2c) \zeta^3 \]
\[ + (c - d \sqrt{m} + 2a + 2c) \zeta^2 + (a - b \sqrt{m} + 2a + 2c) \zeta \in \mathbb{Q}(\sqrt{m}, \zeta). \]
Using Maple we find that

\[ g(x) = x^4 + (10c + 10a)x^3 + (5b^2m + 5d^2m + 80ac + 35a^2 + 35c^2)x^2 \\
+ (30cd^2m + 50c^3 + 200a^2c - 20bcdm + 30ab^2m + 20ad^2m \\
+ 20b^2cm + 200ac^2 + 50a^3 + 20abdm)x - 10b^3dm^2 + 150a^3c \\
+ 25a^2d^2m + 25b^2c^2m - 5b^2d^2m^2 + 275a^2c^2 + 25c^4 + 10bd^3m^2 \\
+ 50acd^2m - 50bc^2dm + 150ac^3 + 50a^2bdm + 50c^2d^2m + 5d^4m^2 \\
+ 25a^4 + 5b^4m^2 + 50a^2b^2m + 50ab^2cm. \]  \tag{13}

The roots of \( g(x) \) are (again using Maple)

\[
\begin{align*}
- \frac{5}{2} a - \frac{5}{2} c + \frac{1}{2} (-a + c) \sqrt{5} + \frac{1}{2} \sqrt{m(10(b^2 + d^2) - (2b^2 + 8bd - 2d^2) \sqrt{5})}, \\
- \frac{5}{2} a - \frac{5}{2} c + \frac{1}{2} (-a + c) \sqrt{5} - \frac{1}{2} \sqrt{m(10(b^2 + d^2) - (2b^2 + 8bd - 2d^2) \sqrt{5})}, \\
- \frac{5}{2} a - \frac{5}{2} c - \frac{1}{2} (-a + c) \sqrt{5} + \frac{1}{2} \sqrt{m(10(b^2 + d^2) + (2b^2 + 8bd - 2d^2) \sqrt{5})}, \\
- \frac{5}{2} a - \frac{5}{2} c - \frac{1}{2} (-a + c) \sqrt{5} - \frac{1}{2} \sqrt{m(10(b^2 + d^2) + (2b^2 + 8bd - 2d^2) \sqrt{5})}.
\end{align*}
\tag{14}
\]

The quantities under the radicals are \( X + Y \sqrt{5} \) and \( X - Y \sqrt{5} \), where

\[ X = -10m(b^2 + d^2), \quad Y = m(2b^2 + 8bd - 2d^2). \] \tag{15}

As

\[ X^2 - 5Y^2 = 5m^2(4b^2 - 4bd - 4d^2)^2, \] \tag{16}

the roots of \( g(x) \) belong to the cyclic quartic field \( \mathbb{Q}(\sqrt{X \pm Y \sqrt{5}}) \) [2, Theorem 1, page 134]. Further

\[ X + Y \sqrt{5} = (-10 + 2\sqrt{5})m \left( \frac{2b - d - d \sqrt{5}}{2} \right)^2 \] \tag{17}

so that (as \( b \neq 0 \) and \( d \neq 0 \))

\[ \mathbb{Q} \left( \sqrt{X + Y \sqrt{5}} \right) = \mathbb{Q} \left( \sqrt{(-10 + 2\sqrt{5})m} \right) = \mathbb{Q} \left( \sqrt{-m(5 + 2\sqrt{5})} \right), \] \tag{18}

as \((-10 + 2\sqrt{5})(-5 - 2\sqrt{5}) = (5 + \sqrt{5})^2\).

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References


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