Some related fixed point theorems for set-valued mappings on two complete and compact uniform spaces are proved.

2000 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction. Let \((X, \mathcal{U}_1)\) and \((Y, \mathcal{U}_2)\) be uniform spaces. Families \(\{d_i^1 : i \in I\}\) of pseudometrics on \(X, Y\), respectively, are called associated families for uniformities \(\mathcal{U}_1, \mathcal{U}_2\), respectively, if families

\[
\beta_1 = \{V_1(i, r) : i \in I, r > 0\},
\]

\[
\beta_2 = \{V_2(i, r) : i \in I, r > 0\},
\]

are subbases for the uniformities \(\mathcal{U}_1, \mathcal{U}_2\), respectively. We may assume that \(\beta_1, \beta_2\) themselves are a base by adjoining finite intersections of members of \(\beta_1, \beta_2\), if necessary. The corresponding families of pseudometrics are called an augmented associated families for \(\mathcal{U}_1, \mathcal{U}_2\). An associated family for \(\mathcal{U}_1, \mathcal{U}_2\) will be denoted by \(\mathcal{D}_1, \mathcal{D}_2\), respectively. For details, the reader is referred to \([1, 4, 5, 6, 7, 8, 9, 10, 11]\).

Let \(A, B\) be a nonempty subset of a uniform space \(X, Y\), respectively. Define

\[
P_1^* (A) = \sup \{d_1^1(x, x') : x, x' \in A, i \in I\},
\]

\[
P_2^* (B) = \sup \{d_2^2(y, y') : y, y' \in B, i \in I\},
\]

where \(\{d_1^1(x, x') : x, x' \in A, i \in I\} = P_1^*\), \(\{d_2^2(y, y') : y, y' \in B, i \in I\} = P_2^*\). Then, \(P_1^* (A), P_2^* (B)\) are called an augmented diameter of \(A, B\). Further, \(A, B\) are said to be \(P_1^* (A) < \infty, P_2^* (B) < \infty\). Let

\[
2^X = \{A : A \text{ is a nonempty } P_1^* \text{-bounded subset of } X\},
\]

\[
2^Y = \{B : B \text{ is a nonempty } P_2^* \text{-bounded subset of } Y\}.
\]
For each \( i \in I \) and \( A_1,A_2 \in 2^X \), \( B_1,B_2 \in 2^Y \), define
\[
\delta_1^i(A_1,A_2) = \sup \{d_1^i(x,x') : x \in A_1, x' \in A_2\}, \\
\delta_2^i(B_1,B_2) = \sup \{d_2^i(y,y') : y \in B_1, y' \in B_2\}.
\] (1.5)

Let \((X,\mathcal{U}_1)\) and \((X,\mathcal{U}_2)\) be uniform spaces and let \( U_1 \in \mathcal{U}_1 \) and \( U_2 \in \mathcal{U}_2 \) be arbitrary entourages. For each \( A \in 2^X \), \( B \in 2^Y \), define
\[
U_1[A] = \{x' \in X : (x,x') \in U_1 \text{ for some } x \in A\}, \\
U_2[B] = \{y' \in Y : (y,y') \in U_2 \text{ for some } y \in B\}.
\] (1.6)

The uniformities \( 2^\mathcal{U}_1 \) on \( 2^X \) and \( 2^\mathcal{U}_2 \) on \( 2^Y \) are defined by bases
\[
2^\mathcal{U}_1 = \{ \tilde{U}_1 : U_1 \in \mathcal{U}_1 \}, \quad 2^\mathcal{U}_2 = \{ \tilde{U}_2 : U_2 \in \mathcal{U}_2 \},
\] (1.7)
where
\[
\tilde{U}_1 = \{(A_1,A_2) \in 2^X \times 2^X : A_1 \times A_2 \subset U_1 \} \cup \Delta, \\
\tilde{U}_2 = \{(B_1,B_2) \in 2^Y \times 2^Y : B_1 \times B_2 \subset U_2 \} \cup \Delta,
\] (1.8)
where \( \Delta \) denotes the diagonal of \( X \times X \) and \( Y \times Y \).

The augmented associated families \( P_1^*, P_2^* \) also induce uniformities \( \mathcal{U}_1^* \) on \( 2^X \), \( \mathcal{U}_2^* \) on \( 2^Y \) defined by bases
\[
\beta_1^* = \{ V_1^*(i,r) : i \in I, r > 0 \}, \\
\beta_2^* = \{ V_2^*(i,r) : i \in I, r > 0 \},
\] (1.9)
where
\[
V_1^*(i,r) = \{(A_1,A_2) : A_1,A_2 \in 2^X : \delta_1^i(A_1,A_2) < r\} \cup \Delta, \\
V_2^*(i,r) = \{(B_1,B_2) : B_1,B_2 \in 2^Y : \delta_2^i(B_1,B_2) < r\} \cup \Delta.
\] (1.10)

Uniformities \( 2^\mathcal{U}_1 \) and \( \mathcal{U}_1^* \) on \( 2^X \) are uniformly isomorphic and uniformities \( 2^\mathcal{U}_2 \) and \( \mathcal{U}_2^* \) on \( 2^Y \) are uniformly isomorphic. The space \((2^X,\mathcal{U}_1^*)\) is thus a uniform space called the hyperspace of \((X,\mathcal{U}_1)\). The \((2^Y,\mathcal{U}_2^*)\) is also a uniform space called the hyperspace of \((Y,\mathcal{U}_2)\).

Now, let \( \{A_n : n = 1,2,...\} \) be a sequence of nonempty subsets of uniform space \((X,\mathcal{U})\). We say that sequence \( \{A_n\} \) converges to subset \( A \) of \( X \) if
(i) each point in \( a \) in \( A \) is the limit of a convergent sequence \( \{a_n\} \), where \( a_n \) is in \( A_n \) for \( n = 1,2,... \),
(ii) for arbitrary \( \varepsilon > 0 \), there exists an integer \( N \) such that \( A_n \subseteq A_\varepsilon \) for \( n > N \), where
\[
A_\varepsilon = \bigcup_{x \in A} U(x) = \{y \in X : d_i(x,y) < \varepsilon \text{ for some } x \in A, i \in I\}.
\] (1.11)

\( A \) is then said to be a limit of the sequence \( \{A_n\} \).

It follows easily from the definition that if \( A \) is the limit of a sequence \( \{A_n\} \), then \( A \) is closed.
Lemma 1.1. If \( \{A_n\} \) and \( \{B_n\} \) are sequences of bounded, nonempty subsets of a complete uniform space \((X, \mathcal{U})\) which converge to the bounded subsets \(A\) and \(B\), respectively, then sequence \(\{\delta_i(A_n,B_n)\}\) converges to \(\delta_i(A,B)\).

Proof. For arbitrary \(\varepsilon > 0\), there exists an integer \(N\) such that
\[
\delta_i(A_n,B_n) \leq \delta_i(A_\varepsilon,B_\varepsilon) = \sup \{d_i(a',b') : a' \in A_\varepsilon, b' \in B_\varepsilon\} \tag{1.12}
\]
for \(n > N\). Now, for each \(a'\) in \(A_\varepsilon\) and \(b'\) in \(B_\varepsilon\), we can find \(a\) in \(A\) and \(b\) in \(B\) with \(d_i(a',a) < \varepsilon, d_i(b',b) < \varepsilon\), and so
\[
d_i(a',b') \leq d_i(a',a) + d_i(a,b) \tag{1.13}
\]
\[
\leq d_i(a,b) + 2\varepsilon.
\]
It follows that
\[
\delta_i(A_n,B_n) < \sup \{d_i(a,b) : a \in A, b \in B\} + 2\varepsilon = \delta_i(A,B) + 2\varepsilon \tag{1.14}
\]
for \(n > N\). Further, there exists an integer \(N'\) such that for each \(a\) in \(A\) and \(b\) in \(B\) we can find \(a_n\) in \(A_n\) and \(b_n\) in \(B_n\) with
\[
d_i(a,a_n) < \varepsilon, \quad d_i(b,b_n) < \varepsilon \tag{1.15}
\]
for \(n > N'\), and so
\[
d_i(a,b) \leq d_i(a,a_n) + d_i(a_n,b) \leq d_i(a,a_n) + d_i(a_n,b_n) + d_i(b_n,b) \tag{1.16}
\]
\[
< d_i(a_n,b_n) + 2\varepsilon.
\]
It follows that
\[
\delta_i(A,B) = \sup \{d_i(a,b) : a \in A, b \in B\} \\
\leq \sup \{d_i(a_n,b_n) : a_n \in A_n, b_n \in B_n\} + 2\varepsilon \tag{1.17}
\]
\[
= \delta_i(A_n,B_n) + 2\varepsilon
\]
for \(n > N'\). The result of the lemma follows from inequalities (1.14) and (1.17).

Remark 1.2. If we replace the uniform space \((X, \mathcal{U})\) in Lemma 1.1 by a metric space (i.e., a metrizable uniform space), then the result of the second author [2] will follow as special case of our result.

Theorem 1.3. Let \((X, \mathcal{U}_1)\) and \((Y, \mathcal{U}_2)\) be complete Hausdorff uniform spaces defined by \(\{d_1^i, i \in I\} = P_1^*, \{d_2^i, i \in I\} = P_2^*\), and \((2^X, \mathcal{U}_1^*), (2^Y, \mathcal{U}_2^*)\) hyperspaces, let \(F : X \to 2^Y\) and \(G : Y \to 2^X\) satisfy inequalities
\[
\delta_1^i(GFx,GFx') \leq c_i \max \{d_1^i(x,x'), \delta_1^i(x,GFx), \delta_1^i(x',GFx'), \delta_2^i(Fx,Fx')\}, \tag{1.18}
\]
\[
\delta_2^i(FGY,FGY') \leq c_i \max \{d_2^i(y,y'), \delta_2^i(y,FGy), \delta_2^i(y',FGy'), \delta_1^i(Gy,Gy')\}
\]
for all \( i \in I \) and \( x, x' \in X, y, y' \in Y \), where \( 0 \leq c_i < 1 \). If \( F \) is continuous, then \( GF \) has a unique fixed point \( z \) in \( X \) and \( FG \) has a unique fixed point \( w \) in \( Y \). Further, \( Fz = \{ w \} \) and \( Gu = \{ z \} \).

**Proof.** Let \( x_1 \) be an arbitrary point in \( X \). Define sequences \( \{ x_n \} \) and \( \{ y_n \} \) in \( X \) and \( Y \), respectively, as follows. Choose a point \( y_1 \) in \( FX_1 \) and then a point \( x_1 \) in \( Gy_1 \). In general, having chosen \( x_n \) in \( X \) and \( y_n \) in \( Y \), choose \( x_{n+1} \) in \( G y_n \) and then \( y_{n+1} \) in \( F x_{n+1} \) for \( n = 1, 2, \ldots \).

Let \( U_1 \in \mathcal{U}_1 \) be an arbitrary entourage. Since \( \beta_1 \) is a base for \( \mathcal{U}_1 \), there exists \( V_1(i, r) \in \beta_1 \) such that \( V_1(i, r) \subseteq U_1 \). We have

\[
d_1^1(x_{n+1}, x_{n+2}) \\ \leq c_i \max \left\{ d_1^1(x_n, x_{n+1}), d_1^1(x_n, G F x_n), d_1^1(x_{n+1}, G F x_{n+1}), d_2^1(F x_n, F x_{n+1}) \right\}
\]

\[
\leq c_i \max \left\{ d_1^1(G F x_{n-1}, G F x_n), d_1^1(G F x_n, G F x_{n+1}), d_2^1(F x_{n-1}, F x_n) \right\}
\]

\[
= c_i \max \left\{ d_1^1(G F x_{n-1}, G F x_n), d_2^1(F x_{n-1}, F x_n) \right\}
\]

and, similarly let \( U_2 \in \mathcal{U}_2 \) be an arbitrary entourage. Since \( \beta_2 \) is a base for \( \mathcal{U}_2 \), there exists \( V_2(i, r) \in \beta_2 \) such that \( V_2(i, r) \subseteq U_2 \). We have

\[
d_2^1(y_{n+1}, y_{n+2}) \leq d_2^1(G F y_{n-1}, G F y_n) \leq c_i \max \left\{ d_2^1(G F y_{n-1}, G F y_n), d_1^1(G y_n, G y_{n+1}) \right\}
\]

(1.20)

It follows that

\[
d_1^1(x_n, x_{n+m}) \leq d_1^1(x_n, x_{n+1}) + d_1^1(x_{n+1}, x_{n+2}) + \cdots + d_1^1(x_{n+m-1}, x_{n+m}) \leq d_1^1(G F x_{n-1}, G F x_n) + \cdots + d_1^1(G F x_{n+m-2}, G F x_{n+m-1}) \leq c_i \max \left\{ d_1^1(G F x_{n-2}, G F x_{n-1}), d_2^1(F x_{n-1}, F x_n) \right\} + \cdots + c_i \max \left\{ d_1^1(G F x_{n+m-3}, G F x_{n+m-2}), d_2^1(F x_{n+m-2}, F x_{n+m-1}) \right\} \leq (c_i^n + c_i^{n+1} + \cdots + c_i^{n+m-1}) d_1^1(x_1, G F x_1)
\]

(1.21)

for \( n \) greater than some \( N \). Since \( c_i < 1 \), it follows that there exists \( p \) such that \( d_1^1(x_n, x_m) < r \) and hence \( (x_n, x_m) \in U_1 \) for all \( n, m \geq p \). Therefore, sequence \( \{ x_n \} \) is Cauchy sequence in the \( d_1^1 \)-uniformity on \( X \).

Let \( S_p = \{ x_n : n \geq p \} \) for all positive integers \( p \) and let \( \mathcal{B}_1 \) be the filter basis \( \{ S_p : p = 1, 2, \ldots \} \). Then, since \( \{ x_n \} \) is a \( d_1^1 \)-Cauchy sequence for each \( i \in I \), it is easy to see that the filter basis \( \mathcal{B}_1 \) is a Cauchy filter in the uniform space \( (X, \mathcal{U}_1) \). To see this, we first note that family \( \{ V_1(i, r) : i \in I, r > 0 \} \) is a base for \( \mathcal{U}_1 \) as \( P_i^* = \{ d_1^1 : i \in I \} \). Now, since \( \{ x_n \} \) is a \( d_1^1 \)-Cauchy sequence in \( X \), there exists a positive integer \( p \) such that \( d_1^1(x_n, x_m) < r \) for \( m \geq p, n \geq p \). This implies that \( S_p \times S_p \subseteq V_1(i, r) \). Thus, given any \( U_1 \in \mathcal{U}_1 \), we can find an \( S_p \in \mathcal{B}_1 \) such that \( S_p \times S_p \subseteq U_1 \). Hence, \( \mathcal{B}_1 \) is a Cauchy filter in \( (X, \mathcal{U}_1) \). Since \( (X, \mathcal{U}_1) \) is a complete Hausdorff space, the Cauchy filter \( \mathcal{B}_1 = \{ S_p \} \)
converges to a unique point $z \in X$. Similarly, the Cauchy filter $\mathcal{B}_2 = \{S_k\}$ converges to a unique point $w \in Y$.

Further,

$$\delta^i_1(z, GF S_p) \leq d^i_1(z, S_{m+1}) + \delta^i_1(S_{m+1}, GF S_p) \leq d^i_1(z, S_{m+1}) + \delta^i_1(GFS_m, GF S_p)$$

(1.22)

since $S_{m+1} \subseteq GFS_m$. Thus, on using inequality (1.20), we have

$$\delta^i_1(z, GF S_p) \leq d^i_1(z, S_{m+1}) + \varepsilon$$

(1.23)

for $n, m \geq p$. Letting $m$ tend to infinity, it follows that

$$\delta^i_1(z, GF S_p) < \varepsilon$$

(1.24)

for $n > p$, and so

$$\lim_{n \to \infty} GF S_p = \{z\}$$

(1.25)

since $\varepsilon$ is arbitrary. Similarly,

$$\lim_{n \to \infty} FG S_k = \{w\} = \lim_{n \to \infty} FS_p$$

(1.26)

since $S_{k+1} \subseteq GS_k$. Using the continuity of $F$, we see that

$$\lim_{p \to \infty} FS_p = Fz = \{w\}.$$  

(1.27)

Now, let $W \in \mathcal{U}_1$ be an arbitrary entourage. Since $\beta_1$ is a base for $\mathcal{U}_1$, there exists $V_1(j, t) \in \beta_1$ such that $V_1(j, t) \subseteq W$. Using inequality (1.14), we now have

$$\delta^i_1(GFS_p, GF z) \leq c_i \max \{d^i_1(S_p, z), \delta^i_1(S_p, GFS_p), \delta^i_1(z, GF z), \delta^i_2(Fz, FS_p)\}.$$  

(1.28)

Letting $p$ tend to infinity and using (1.24) and (1.26), we have

$$\delta^i_1(z, GF z) \leq c_i \delta^i_1(z, GF z).$$

(1.29)

Since $c_i < 1$, we have $\delta^i_1(z, GF z) = 0 < t$. Hence, $(z, GF z) \in V_1(j, t) \subseteq W$. Again, since $W$ is arbitrary and $X$ is Hausdorff, we must have $GF z = \{z\}$, proving that $z$ is a fixed point of $GF$.

Further, using (1.26), we have

$$FG w = FGF z = w,$$

(1.30)

proving that $w$ is a fixed point of $FG$. 

Now, suppose that $GF$ has a second fixed point $z'$. Then, using inequalities (1.18), we have

$$\begin{align*}
\delta_i^1(z', GFz') &\leq \delta_i^1(GFz', GFz') \\
&\leq c_i \max \{d_i^1(z', z'), \delta_i^1(z', GFz'), \delta_i^2(Fz', Fz')\} \\
&\leq c_i \delta_i^1(Fz', Fz') \leq c_i \delta_i^1(Fz', FGFz') \leq c_i \delta_i^1(FGFz', FGFz') \\
&\leq c_i^2 \max \{\delta_i^2(Fz', FGFz'), \delta_i^2(Fz', FGFz'), \delta_i^1(GFz', GFz')\} \\
&\leq c_i^2 \delta_i^1(GFz', GFz'),
\end{align*}$$

(1.31)

and so $Fz'$ is a singleton and $GFz' = \{z'\}$, since $c_i < 1$. Thus,

$$\begin{align*}
\delta_i^1(z, z') &\leq \delta_i^1(GFz, GFz') \\
&\leq c_i \max \{d_i^1(z, z'), \delta_i^1(z, GFz), \delta_i^1(z', GFz'), \delta_i^2(Fz, Fz')\}.
\end{align*}$$

(1.32)

But

$$\begin{align*}
d_i^2(Fz, Fz') &\leq \delta_i^2(FGFz, FGFz') \\
&\leq c_i \max \{\delta_i^2(Fz, Fz'), \delta_i^2(Fz, FGFz), \delta_i^2(Fz', FGFz'), \delta_i^1(GFz, GFz')\} \\
&= c_i \max \{d_i^2(Fz, Fz'), d_i^2(Fz, Fz), d_i^2(Fz', Fz'), d_1^1(z, z')\} \\
&= c_i d_i^1(z, z'),
\end{align*}$$

(1.33)

and so

$$d_i^1(z, z') \leq c_i^2 d_i^1(z, z').$$

(1.34)

Since $c_i < 1$, the uniqueness of $z$ follows.

Similarly, $w$ is the unique fixed point of $FG$. This completes the proof of the theorem. \hfill \square

If we let $F$ be a single-valued mapping $T$ of $X$ into $Y$ and $G$ a single-valued mapping $S$ of $Y$ into $X$, we obtain the following result.

**Corollary 1.4.** Let $(X, \mathcal{U}_1)$ and $(Y, \mathcal{U}_2)$ be complete Hausdorff uniform spaces. If $T$ is a continuous mapping of $X$ into $Y$ and $S$ is a mapping of $Y$ into $X$ satisfying the inequalities

$$\begin{align*}
d_i^1(STx, STx') &\leq c_i \max \{d_i^1(x, x'), d_i^1(x, STx), d_i^1(x', STx'), d_i^2(Tx, Tx')\}, \\
d_i^2(TSy, TSy') &\leq c_i \max \{d_i^2(y, y'), d_i^2(y, TSy), d_i^2(y', TSy'), d_i^1(Sy, Sy')\}
\end{align*}$$

(1.35)

for all $x, x' \in X$ and $y, y' \in Y$, $i \in I$ where $0 \leq c_i < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$. 
Theorem 1.5. Let \((X, \mathcal{U}_1)\) and \((Y, \mathcal{U}_2)\) be compact uniform spaces defined by \(\{d_i^1 : i \in I\} = P^*_1\) and \(\{d_i^2 : i \in I\} = P^*_2\), and, \((2^X, \mathcal{U}^*_1)\) and \((2^Y, \mathcal{U}^*_2)\) hyperspaces. If \(F\) is a continuous mapping of \(X\) into \(2^X\) and \(G\) is a continuous mapping of \(Y\) into \(2^X\) satisfying the inequalities

\[
\delta_1^1(GFx, GFx') < \max \{d_1^1(x, x'), \delta_1^1(x, GFx), \delta_1^1(x', GFx'), \delta_2^2(Fx, Fx')\},
\]
\[
\delta_2^2(FGy, FGy') < \max \{d_2^2(y, y'), \delta_2^2(y, FGy), \delta_2^2(y', FGy'), \delta_1^1(Gy, Gy')\}
\]

for all \(x, x' \in X\) and \(y, y' \in Y\), \(i \in I\) for which the right-hand sides of the inequalities are positive, then, \(FG\) has a unique fixed point \(z \in X\) and \(GF\) has a unique fixed point \(w \in Y\). Further, \(FGz = \{z\}\) and \(GFw = \{w\}\).

Proof. We denote the right-hand sides of inequalities \((1.35)\) by \(h(x, x')\) and \(k(y, y')\), respectively. First of all, suppose that \(h(x, x') \neq 0\) for all \(x, x' \in X\) and \(k(y, y') \neq 0\) for all \(y, y' \in Y\). Define the real-valued function \(f(x, x')\) on \(X \times X\) by

\[
f(x, x') = \frac{\delta_1^1(GFx, GFx')}{h(x, x')}.\]

Then, if \(\{(x_n, x'_n)\}\) is an arbitrary sequence in \(X \times X\) converging to \((x, x')\), it follows from the lemma and the continuity of \(F\) and \(G\) that the sequence \(\{f(x_n, x'_n)\}\) converges to \(f(x, x')\). The function \(f\) is therefore a continuous function defined on the compact uniform space \(X \times X\) and so achieves its maximum value \(c_1^1 < 1\).

Thus,

\[
\delta_1^1(GFx, GFx') \leq c_1^1 \max \{d_1^1(x, x'), \delta_1^1(x, GFx), \delta_1^1(x', GFx'), \delta_2^2(Fx, Fx')\}
\]

for all \(x, x' \in X, i \in I\).

Similarly, there exists \(c_2^1 < 1\) such that

\[
\delta_2^2(FGy, FGy') \leq c_2^1 \max \{d_2^2(y, y'), \delta_2^2(y, FGy), \delta_2^2(y', FGy'), \delta_1^1(Gy, Gy')\}
\]

for all \(y, y' \in Y, i \in I\). It follows that the conditions of Theorem 1.3 are satisfied with \(c_i = \max\{c_1^i, c_2^i\}\) and so, once again there exists \(z\) in \(X\) and \(w\) in \(Y\) such that \(GFz = \{z\}\) and \(FGw = \{w\}\).

Now, suppose that \(h(x, x') = 0\) for some \(x, x' \in X\). Then, \(GFx = GFx' = \{x\} = \{x'\}\) is a singleton \(\{w\}\). It follows that \(z\) is a fixed point of \(GF\) and \(GFz = \{z\}\). Further,

\[
FGw = FGFz = Fz = \{w\}
\]

and so \(w\) is a fixed point of \(FG\).

It follows similarly that if \(k(y, y') = 0\) for some \(y, y' \in Y\), then again \(GF\) has a fixed point \(z\) and \(FG\) has a fixed point \(w\).
Now, we suppose that $GF$ has a second fixed point $z'$ in $X$ so that $z'$ is in $GFz'$. Then, on using inequalities (1.36), we have, on assuming that $\delta_i^1(Fz', Fz') \neq 0$ for each $i \in I$,

$$\delta_i^1(z', GFz') \leq \delta_i^1(GFz', GFz')$$

$$< \max \{ \delta_i^1(z', z'), \delta_i^1(z', GFz'), \delta_i^2(Fz', Fz') \}$$

$$= \delta_i^2(Fz', Fz') \leq \delta_i^2(Fz', FGFz') \leq \delta_i^2(GFGz', FGFz')$$

$$< \max \{ \delta_i^2(Fz', Fz'), \delta_i^2(Fz', FGFz'), \delta_i^1(GFz', GFz') \}$$

$$= \delta_i^1(GFz', GFz'), \tag{1.41}$$

a contradiction, and so $Fz'$ is a singleton and $GFz' = \{ z' \}$. Thus, if $z \neq z'$

$$d_i^1(z, z') = \delta_i^1(GFz, GFz')$$

$$< \max \{ d_i^1(z, z'), \delta_i^1(z, GFz), \delta_i^1(z', GFz'), \delta_i^2(Fz, Fz') \}$$

$$= d_i^2(Fz, Fz'), \tag{1.42}$$

But if $Fz \neq Fz'$, we have

$$d_i^2(Fz, Fz') \leq \delta_i^2(GFGz, FGFz')$$

$$< \max \{ \delta_i^2(Fz, Fz'), \delta_i^2(Fz, FGFz), \delta_i^2(Fz', FGFz'), \delta_i^1(GFz, GFz') \}$$

$$= \max \{ \delta_i^2(Fz, Fz'), d_i^2(Fz, Fz), d_i^2(Fz', Fz'), d_i^1(z, z') \}$$

$$= d_i(z, z'), \tag{1.43}$$

and so

$$d_i(z, z') < d_i(z, z'), \tag{1.44}$$

a contradiction. The uniqueness of $z$ follows.

Similarly, $w$ is the unique fixed point of $FG$. This completes the proof of the theorem.

If we let $F$ be a single-valued mapping $T$ of $X$ into $Y$ and $G$ a single-valued mapping of $Y$ into $X$, we obtain the following result.

**Corollary 1.6.** Let $(X, \mathcal{U}_1)$ and $(Y, \mathcal{U}_2)$ be compact Hausdorff uniform spaces. If $T$ is a continuous mapping of $X$ into $Y$ and $S$ is a continuous mapping of $Y$ into $X$ satisfying the inequalities

$$d_i^1(STx, STx') < \max \{ d_i^1(x, x'), d_i^1(x, STx), d_i^1(x', STx'), d_i^1(Tx, Tx') \},$$

$$d_i^2(TSy, TSy') < \max \{ d_i^2(y, y'), d_i^2(y, TSy), d_i^2(y', TSy'), d_i^2(Sy, Sy') \}, \tag{1.45}$$

for all $x, x' \in X$ and $y, y' \in Y, i \in I$ for which the right-hand sides of the inequalities are positive, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$. 

Remark 1.7. If we replace the uniform spaces $(X, ℱ_1)$ and $(Y, ℱ_2)$ in Theorems 1.3 and 1.5 and Corollaries 1.4 and 1.6, by a metric space (i.e., a metrizable uniform space), then the results of the authors [3] will follow as special cases of our results.

Acknowledgment. This work has been supported by Gazi University project no. 05/2003-01, Turkey.

References


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