We study the action of \( A \) on \( f \in L^2(\mathbb{R}) \) and on its wavelet coefficients, where \( A = (a_{lmjk})_{lmjk} \) is a double infinite matrix. We find the frame condition for \( A \)-transform of \( f \in L^2(\mathbb{R}) \) whose wavelet series expansion is known.

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1. Introduction. The notation of frame goes back to Duffin and Schaeffer [7] in the early 1950s to deal with the problems in nonharmonic Fourier series. There has been renewed interest in the subject related to its role in wavelet theory. For a glance of the recent development and work on frames and related topics, see [3, 4, 5, 6, 9]. In this note, we will use the regular double infinite matrices (see [9, 10]) to obtain the frame conditions and wavelet coefficients.

2. Notations and known results. \( \mathbb{N} \) is the set of positive integers, \( \mathbb{Z} \) is the set of integers, \( \mathbb{R} \) is the set of real numbers. The space \( L^2(\mathbb{R}) \) of measurable function \( f \) is defined on the real line \( \mathbb{R} \), that satisfies

\[
\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty. \tag{2.1}
\]

The inner product of two square integrable functions \( f, g \in L^2(\mathbb{R}) \) is defined as

\[
\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) dx,
\]

\[
\|f\|^2 = (f, f)^{1/2}. \tag{2.2}
\]

Every function \( f \in L^2(\mathbb{R}) \) can be written as

\[
f(x) = \sum_{j,k \in \mathbb{Z}} C_{j,k} \psi_{j,k}(x). \tag{2.3}
\]

This series representation of \( f \) is called wavelet series. Analogous to the notation of Fourier coefficients, the wavelet coefficients \( C_{j,k} \) are given by

\[
C_{j,k} = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx = \langle f, \psi_{j,k} \rangle,
\]

\[
\psi_{j,k} = 2^{j/2} \psi(2^j x - k). \tag{2.4}
\]
Now, if we define an integral transform
\[(W_\psi f)(b,a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x) \psi \left( \frac{x-b}{a} \right) dx, \quad f \in L^2(\mathbb{R}), \quad (2.5)\]
then the wavelet coefficients become
\[C_{j,k} = (W_\psi f) \left( \frac{k}{2^j}, \frac{1}{2^j} \right), \quad (2.6)\]

A sequence \(\{x_n\}\) in a Hilbert space \(H\) is a frame if there exist constants \(c_1\) and \(c_2\), \(0 < c_1 \leq c_2 < \infty\), such that
\[c_1 \|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, x_n \rangle|^2 \leq c_2 \|f\|^2, \quad (2.7)\]
for all \(f \in H\). The supremum of all such numbers \(c_1\) and infimum of all such numbers \(c_2\) are called the frame bounds of the frame. The frame is called tight frame when \(c_1 = c_2\) and is called normalized tight frame when \(c_1 = c_2 = 1\). Any orthonormal basis in a Hilbert space \(H\) is a normalized tight frame. The connection between frames and numerically stable reconstruction from discretized wavelet was pointed out by Grossmann et al. [8]. In 1985, they defined that a wavelet function \(\psi \in L^2(\mathbb{R})\), constitutes a frame with frame bounds \(c_1\) and \(c_2\), if any \(f \in L^2(\mathbb{R})\) such that
\[c_1 \|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 \leq c_2 \|f\|^2. \quad (2.8)\]
Again, it is said to be tight if \(c_1 = c_2\) and is said to be exact if it ceases to be frame by removing any of its elements. There are many examples proposed by Daubechies et al. [6]. For further details, one can refer to [1, 5, 6]. Chui and Shi [2] proved that \(\{\psi_{j,k}\}\) is a frame for \(L^2(\mathbb{R})\) with bounds \(c_1\) and \(c_2\), if for some \(a > 1\) and \(b > 0\), the Fourier transform \(\hat{\psi}\) satisfies
\[c_1 \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^j w)|^2 \leq c_2 \quad \text{a.e.,} \quad (2.9)\]
for some constants \(c_1\) and \(c_2\). By integrating each term in
\[\frac{c_1}{|w|} \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} \frac{|\hat{\psi}(a^j w)|^2}{|w|} \leq \frac{c_2}{|w|} \quad (2.10)\]
over \(1 \leq |w| \leq a\), we have
\[2c_1 \log a \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} \int_{1 \leq |w| \leq a} \frac{|\hat{\psi}(a^j w)|^2}{|w|} dw \leq 2c_2 \log a, \quad (2.11)\]
which immediately yields
\[c_1 \leq \frac{1}{2b \log a} \int_{-\infty}^{\infty} \frac{|\hat{\psi}(a^j w)|^2}{|w|} dw \leq c_2. \quad (2.12)\]
The above condition known as compactibility condition was also observed by Daubechies [4] by using techniques from trace class operators. The above constants were given by frame bounds, see [2].

Let \( A = (a_{mnjk}) \) be a double infinite matrix of real numbers. Then, \( A \)-transform of a double sequence \( x = (x_{jk}) \) is

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk} x_{jk},
\]

which is called \( A \)-means or \( A \)-transform of the sequence \( x = (x_{ij}) \). This definition is due to Móricz and Rhoades [9].

A double matrix \( A = (a_{mnjk}) \) is said to be regular (see [10]) if the following conditions hold:

1. \( \lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} a_{mnjk} = 1 \),
2. \( \lim_{m,n \to \infty} \sum_{j=0}^{\infty} |a_{mnjk}| = 0, (k = 0, 1, 2, \ldots) \),
3. \( \lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{mnjk}| = 0, (j = 0, 1, 2, \ldots) \),
4. \( \|A\| = \sup_{m,n>0} \sum_{j,k=0}^{\infty} |a_{m,n}| < \infty \).

Either of conditions (ii) and (iii) implies that

\[
\lim_{m,n \to \infty} a_{mnjk} = 0.
\]

In this note, we establish the frame condition by using \( A \)-transform of nonnegative regular matrix, also we find action of the matrix \( A \) on wavelet coefficients.

3. Main results. In this section, we prove the following theorems.

**Theorem 3.1.** Let \( A = (a_{iljk}) \) be a double nonnegative regular matrix. If

\[
f(x) = \sum_{j,k \in Z} C_{j,k} \psi_{j,k}(x)
\]

is a wavelet expansion of \( f \in L^2(\mathbb{R}) \) with wavelet coefficients

\[
C_{j,k} = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx = \langle f, \psi_{j,k} \rangle,
\]

then the frame condition for \( A \)-transform of \( f \in L^2(\mathbb{R}) \) is

\[
c_1 \|f\|^2 \leq \sum_{i,l \in Z} |\langle Af, \psi_{i,l} \rangle|^2 \leq c_2 \|f\|^2,
\]

where \( Af \) is the \( A \)-transform of \( f \) and \( 0 < c_1 \leq c_2 < \infty \).

**Theorem 3.2.** If \( C_{j,k} \) are the wavelet coefficients of \( f \in L^2(\mathbb{R}) \), that is, \( C_{j,k} = \langle f, \psi_{j,k} \rangle \), then the \( d_{l,m} \) are the wavelet coefficients of \( Af \), where \( \{d_{l,m}\} \) is defined as the \( A \)-transform of \( \{C_{j,k}\} \) by

\[
d_{l,m} = \sum_{j,k=-\infty}^{\infty} a_{lmjk} C_{jk}.
\]
**Theorem 3.3.** Let $A = (a_{lm})$ be a double nonnegative matrix whose elements are $\langle \psi_{j,k}, \psi_{l,m} \rangle$. Then, $\{ \psi_{j,k} \}$ constitutes a frame of $L^2(\mathbb{R})$ if and only if $\{ \psi_{l,m} \}$ constitutes a frame of $L^2(\mathbb{R})$, where $C_{j,k} = \langle f, \psi_{j,k} \rangle$ and $d_{l,m} = \langle f, \psi_{l,m} \rangle$.

**Proof of Theorem 3.1.** We can write

$$f(x) = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}. \quad (3.5)$$

If we take $A$-transform of $f$, we get

$$Af(x) = \sum_{i,l \in \mathbb{Z}} \langle Af, \psi_{i,l} \rangle \psi_{i,l}, \quad (3.6)$$

and therefore

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq \sum_{i,l \in \mathbb{Z}} \int_{-\infty}^{\infty} |Af(x)|^2 |\overline{\psi_{i,l}(x)}|^2 dx \leq \|A\|^2 \|f\|_2^2 \sum_{i,l \in \mathbb{Z}} \|\psi_{i,l}\|_2^2. \quad (3.7)$$

Since $A$ is regular matrix and $\|\psi_{i,l}\|_2 = 1$, therefore

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq c_2 \|f\|_2^2, \quad (3.8)$$

where $c_2$ is positive constant.

Now, for any arbitrarily $f \in L^2(\mathbb{R})$, define

$$\tilde{f} = \left[ \sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \right]^{-1/2} f. \quad (3.9)$$

Clearly,

$$\langle A\tilde{f}, \psi_{i,l} \rangle = \left[ \sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \right]^{-1/2} \langle Af, \psi_{i,l} \rangle, \quad (3.10)$$

then

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq 1. \quad (3.11)$$
Hence, if there exists $\alpha$ a positive constant, then
\[
\|A\hat{f}\|_2^2 \leq \alpha,
\]
\[
\left[ \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2 \right]^{-1} \|Af\|_2^2 \leq \alpha.
\]
(3.12)

Since $A$ is regular, we have
\[
\left[ \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2 \right]^{-1} \|f\|_2^2 \leq \alpha_1 \left( \frac{\alpha}{\|A\|^2} \right),
\]
where $\alpha_1$ is another positive constant. Therefore,
\[
c_1 \|f\|_2^2 \leq \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2,
\]
(3.14)

where $c_1 = \alpha > 0$.

Combining (3.8) and (3.14), we have
\[
c_1 \|f\|_2^2 \leq \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2 \leq c_2 \|f\|_2^2.
\]
(3.15)

This completes the proof. \qed

**Proof of Theorem 3.2.** We can write
\[
\langle Af, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} Af(x) \overline{\psi_{l,m}(x)} dx
\]
\[
= \int_{-\infty}^{\infty} \sum_{j,k=\infty}^{\infty} a_{lmj,k} \psi_{j,k}(x) \overline{\psi_{l,m}(x)} dx.
\]
(3.16)

Now,
\[
\sum_{l,m=\infty}^{\infty} \langle Af, \psi_{l,m} \rangle \psi_{l,m} = \sum_{l,m=\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j,k=\infty}^{\infty} a_{lmj,k} \psi_{j,k}(x) \psi_{l,m}(x) \overline{\psi_{l,m}(x)} dx
\]
\[
= \sum_{l,m=\infty}^{\infty} d_{l,m} \psi_{l,m} \int_{-\infty}^{\infty} \|\psi_{l,m}(x)\|_2^2
\]
\[
= \sum_{l,m=\infty}^{\infty} d_{l,m} \psi_{l,m}.
\]
(3.17)

Therefore,
\[
\sum_{l,m=\infty}^{\infty} d_{l,m} \psi_{l,m} = \sum_{l,m=\infty}^{\infty} \langle Af, \psi_{l,m} \rangle \psi_{l,m}.
\]
(3.18)

This implies that $d_{l,m}$ are wavelet coefficients of $Af$. 
Thus,

\[ d_{l,m} = \langle f, \psi_{l,m} \rangle. \tag{3.19} \]

This completes the proof.

**Proof of Theorem 3.3.** We observe that

\[
\begin{align*}
\alpha_{l,m,j,k} c_{j,k} &= \langle \psi_{j,k}, \psi_{l,m} \rangle \langle f, \psi_{j,k} \rangle \\
&= \int_{-\infty}^{\infty} \psi_{j,k}(x) \overline{\psi_{l,m}(x)} dx \left[ \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx \right] \\
&= \int_{-\infty}^{\infty} f(x) \overline{\psi_{l,m}(x)} dx \left[ \int_{-\infty}^{\infty} \psi_{j,k}(x) \overline{\psi_{j,k}(x)} dx \right] \\
&= \langle f, \psi_{l,m} \rangle,
\end{align*}
\]

that is, \( \alpha_{l,m,j,k} c_{j,k} = d_{l,m} \).

Now,

\[
\begin{align*}
\sum_{l,m} |d_{l,m}|^2 &= \sum_{l,m} |\alpha_{l,m,j,k} c_{j,k}|^2 = \sum_{l,m} |\langle f, \psi_{l,m} \rangle|^2 \\
&= \frac{1}{(2\pi)^2} \sum_{l,m} |\langle \hat{f}, \psi_{l,m} \rangle|^2,
\end{align*}
\]

by Parseval’s formula for trigonometric Fourier series.

Now

\[
\begin{align*}
\frac{1}{(2\pi)^2} \sum_{l,m} \left[ \left| \int_{0}^{2\pi} \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\pi p)} e^{ilmw} dw \right|^2 \\
= p \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{p=\infty}^{\infty} \hat{f}(w + 2\pi p) \overline{\psi(w + 2\pi p)} dw \right|^2
\end{align*}
\]

by Parseval’s formula for trigonometric Fourier series.

Let \( f(w) = \sum_{q=\infty}^{\infty} \hat{f}(w + 2\pi q) \psi(w + 2\pi q) \).
Therefore,
\[
p = \frac{1}{2\pi} \left[ \int_0^{2\pi} \left| \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p)\hat{\psi}(w+2\pi p) \right|^2 \, dw \right]
\]
\[
= \frac{1}{2\pi} \left( \int_0^{2\pi} \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p)\hat{\psi}(w+2\pi p) \, dw \, F(w) \, dw \right)
\]
\[
= \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \hat{f}(w)\hat{\psi}(w)F(w) \, dw \right)
\]
\[
= \frac{1}{2\pi} \left\{ \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(w)\hat{\psi}(w)\hat{f}(w+2\pi q)\hat{\psi}(w+2\pi q) \, dw \right\}
\]
\[
= \frac{1}{2\pi} \left\{ \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{f}(w)|^2 |\hat{\psi}(w)|^2 \, dw \right\}
\]
\[
= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} |\hat{f}(w)|^2 \, dw \right\}
\]
\[
= \|f\|^2_2,
\]
that is,
\[
\sum_{l,m} |d_{lm}|^2 = \|f\|^2_2, \quad f \in L^2(\mathbb{R}). \tag{3.24}
\]

Therefore, for a regular matrix \(A = (a_{lmj,k})\), we have
\[
c_1 \|f\|^2_2 \leq \sum_{l,m} |d_{lm}|^2 \leq c_2 \|f\|^2_2 \tag{3.25}
\]
if and only if
\[
c_1' \|f\|^2_2 \leq \sum_{j,k} |c_{jk}|^2 \leq c_2' \|f\|^2_2, \tag{3.26}
\]
where, \(0 \leq c'_1, c'_2 < \infty\). This completes the proof. □

REFERENCES


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