UNIT-CIRCLE-PRESERVING MAPPINGS

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We prove that if a one-to-one mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) preserves the unit circles, then \( f \) is a linear isometry up to translation.

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1. Introduction. Let \( X \) and \( Y \) be normed spaces. A mapping \( f : X \to Y \) is called an isometry if \( f \) satisfies the equality

\[
\|f(x) - f(y)\| = \|x - y\| \tag{1.1}
\]

for all \( x, y \in X \). A distance \( r > 0 \) is said to be preserved (conserved) by a mapping \( f : X \to Y \) if

\[
\|f(x) - f(y)\| = r \quad \forall x, y \in X \text{ with } \|x - y\| = r. \tag{1.2}
\]

If \( f \) is an isometry, then every distance \( r > 0 \) is conserved by \( f \), and vice versa. We can now raise a question whether each mapping that preserves certain distances is an isometry. Indeed, Aleksandrov [1] had raised a question whether a mapping \( f : X \to X \) preserving a distance \( r > 0 \) is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume \( r = 1 \) when \( X \) is a normed space (see [16]).

Beckman and Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces \( X = \mathbb{R}^n \) (see also [3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 17, 18, 19, 20]).

Theorem 1.1 (Beckman and Quarles). If a mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) preserves a distance \( r > 0 \), then \( f \) is a linear isometry up to translation.

Recently, Zaks [25] proved the rational analogues of the Beckman-Quarles theorem. Indeed, he assumes that \( n = 4k(k + 1) \) for some \( k \geq 1 \) or \( n = 2m^2 - 1 \) for some \( m \geq 3 \), and he proves that if a mapping \( f : \mathbb{Q}^n \to \mathbb{Q}^n \) preserves the unit distance, then \( f \) is an isometry (see also [21, 22, 23, 24]).

It seems interesting to investigate whether the “distance \( r > 0 \)” in the Beckman-Quarles theorem can be replaced by some properties characterized by “geometrical figures” without loss of its validity.

In [9], the first author proved that if a one-to-one mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) maps every regular triangle (quadrilateral or hexagon) of side length \( a > 0 \) onto a figure of
the same type with side length \( b > 0 \), then there exists a linear isometry \( I : \mathbb{R}^n \to \mathbb{R}^n \) up to translation such that

\[
f(x) = \frac{b}{a} I(x).
\]

Furthermore, the first author proved that if a one-to-one mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) maps every unit circle onto a unit circle, then \( f \) is a linear isometry up to translation (see [10]).

In this connection, we will extend the result of [10] to the \( n \)-dimensional cases; more precisely, we prove in this paper that if a one-to-one mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) maps every unit circle onto a unit circle, then \( f \) is a linear isometry up to translation.

2. Preliminaries. We start with any two distinct points \( a \) and \( b \) in \( \mathbb{R}^n \) with the distance between the two less than 2. Let their distance be

\[
2c = 2 \sin \varphi_0 \quad \text{with} \quad 0 < \varphi_0 < \frac{\pi}{2}, \quad 0 < c < 1.
\]

Given such two distinct points whose distance is less than 2, we can choose a coordinate \((y_1, \ldots, y_n)\) for \( \mathbb{R}^n \) such that

\[
a = (0, \ldots, 0, \sin \varphi_0), \quad b = (0, \ldots, 0, -\sin \varphi_0).
\]

Let the \((n - 2)\)-dimensional unit sphere contained in the space orthogonal to the \( y_n \)-direction be

\[
Y = \{(y_1, \ldots, y_{n-1}, 0) \mid y_1^2 + \cdots + y_{n-1}^2 = 1\}.
\]

If we call the center of any unit circle passing through the two points \( a \) and \( b \) \( o' \) and the origin of the coordinate \( o \), then the vector \( oo' \) is perpendicular to the \( y_n \)-axis and its length must be \( \cos \varphi_0 \) and therefore \( oo' \in \tilde{Y} = \cos \varphi_0 Y \), see Figure 2.1. It means that any unit circle passing through the points \( a \) and \( b \) has its center in \( \tilde{Y} = \cos \varphi_0 Y \). Let \( T \) be the set of union of all the unit circles passing through the points \( a \) and \( b \). More precisely, if we define the following set:

\[
T = \{ (\cos \varphi + \cos \varphi_0) y + (0, \ldots, 0, \sin \varphi) \mid y \in Y, \ 0 \leq \varphi < 2\pi \},
\]
then it is clear that this is the set of union of all the unit circles which are centered at \( \cos \varphi_0 \gamma \) for each fixed \( \gamma \in Y \) and which pass through \( a \) and \( b \) when \( \varphi = \pi \mp \varphi_0 \) (see Figure 2.1).

The intersection of \( T \) and the \( \gamma_1 \cdot \gamma_n \) plane consists of two circles, say \( C_1 \) (when \( \gamma_1 = 1 \), i.e., \( \gamma = (1,0,\ldots,0) \)) and \( C_2 \) (when \( \gamma_1 = -1 \), i.e., \( \gamma = (-1,0,\ldots,0) \), see Figure 2.1). In the following contexts, we will consider the cases \( \gamma_1 = 1 \) and \( -1 \) in connection with \( T \) as the circles \( C_1 \) and \( C_2 \), respectively. Call \( S_1 \) the \((n-1)\)-dimensional unit sphere containing the circle \( C_1 \). If we let the center of \( C_1 \) be \( O \) and the center of \( S_1 \) be \( \hat{O} \), then it is obvious that \( O = \hat{O} \).

(To see this, choose any point \( A \in C_1 \) and its antipodal point \( B \) in \( C_1 \). Then, by the definition of the antipodal points that they lie exactly the opposite with respect to the center of the circle \( C_1 \) whose center is at \( O \), and because they are of the same length 1, we have the following condition that
\[
\overrightarrow{OA} = -\overrightarrow{OB}, \quad \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = 2 \overrightarrow{OB}.
\]

On the other hand, we have, since the two points \( A \) and \( B \) lie also on the unit sphere \( S_1 \) with its center at \( \hat{O} \),
\[
2 = |\overrightarrow{AB}| = |\overrightarrow{A\hat{O}} + \overrightarrow{OB}| \leq |\overrightarrow{A\hat{O}}| + |\overrightarrow{OB}| = 1 + 1 = 2.
\]

Therefore, by the Cauchy-Schwarz inequality, \( \overrightarrow{A\hat{O}} \) is a positive multiple of \( \overrightarrow{OB} \), which means \( A\hat{O} = \overrightarrow{OB} \) because their lengths are both 1. So,
\[
\overrightarrow{AB} = \overrightarrow{A\hat{O}} + \overrightarrow{OB} = 2 \overrightarrow{OB},
\]
and therefore \( \hat{O} = O \).

Now, we first show that \( S_1 \) and \( T \) intersect only at \( C_1 \). To make computation simpler we use a new coordinate \( x \) for \( \mathbb{R}^n \), where
\[
x = \gamma - (\cos \varphi_0,0,\ldots,0).
\]

In this coordinate (see Figure 2.2), \( S_1 \) becomes the unit sphere \( S \) centered at the origin,
\[
S_1 = S = \left\{ (x_1,\ldots,x_n) \mid x_1^2 + \cdots + x_n^2 = 1 \right\},
\]
\[
T = \left\{ x = (\cos \varphi + \cos \varphi_0) \gamma + (0,\ldots,0,\sin \varphi) - (\cos \varphi_0,0,\ldots,0) \mid \gamma \in Y, \ 0 \leq \varphi < 2\pi \right\}.
\]

With the help of this coordinate we show the following lemma.

**Lemma 2.1.** \( T \cap S_1 = C_1 \).
**Proof.** If any element in $T$ has distance 1 from the origin of the $x$-coordinate, then we have

$$1 = [(\cos \varphi + \cos \varphi_0) y_1 - \cos \varphi_0]^2 + (\cos \varphi + \cos \varphi_0)^2 y_2^2 + \cdots + (\cos \varphi + \cos \varphi_0)^2 y_{n-1}^2 + \sin^2 \varphi$$

$$= (\cos \varphi + \cos \varphi_0)^2 - 2 \cos \varphi_0 (\cos \varphi + \cos \varphi_0) y_1 + \cos^2 \varphi_0 + \sin^2 \varphi$$

$$= 1 + 2 \cos^2 \varphi_0 (1 - y_1) + 2 \cos \varphi_0 \cos \varphi (1 - y_1).$$

Therefore, we have

$$0 = 2 \cos \varphi_0 (1 - y_1) (\cos \varphi + \cos \varphi_0).$$

With $y_1 = 1$, $T$ in (2.10) represents the unit circle $C_1$ in the $x_1$-$x_n$ plane. If

$$\cos \varphi = -\cos \varphi_0,$$

i.e., $\varphi = \pi \mp \varphi_0,$

then it follows from (2.10) that

$$T = \{x = (-\cos \varphi_0, 0, \ldots, 0, \pm \sin \varphi_0)\} = \{a, b\}$$

which also belong to $C_1$. $\square$

Now, consider, as in **Figure 2.3**, the origin $e$ and $\bar{e} = (-2, 0, \ldots, 0)$ in the $x$-coordinate and the unit circle $C_1$ passing through $e$ and $\bar{e}$ in the $x_1$-$x_n$ plane. Choose a point $d \in C_1$, $d \notin \{e, \bar{e}\}$. We parameterize all the unit circles passing through the points $e$ and $d$. We assume the $x_n$-coordinate of $d$ is negative.

By triangle inequality, the distance between $e$ and $d$ is less than 2, say $2 \sin \varphi_0$, with $0 < \varphi_0 < \pi/2$. Choose a new coordinate $y'$ for $\mathbb{R}^n$ and consider two points

$$e' = (0, \ldots, 0, \sin \varphi_0), \quad d' = (0, \ldots, 0, -\sin \varphi_0),$$

(see **Figure 2.4**).
To get a parameterization of the unit circles passing through e and d, we consider the mapping $M$ defined by

$$
x = My = \begin{bmatrix} \cos \varphi_0 & 0 & \cdots & 0 & \sin \varphi_0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\sin \varphi_0 & 0 & \cdots & 0 & \cos \varphi_0 \end{bmatrix} \left[ y + (\cos \varphi_0, 0, \ldots, 0) \right] - (1, 0, \ldots, 0). \tag{2.16}
$$

This transformation $M$ is an isometry (since it is a composition of a rotation and translations) and sends

$$\{ y = (0, \ldots, 0, \pm \sin \varphi_0) \} = \{ e', d' \} \tag{2.17}$$

to

$$\{ x = (0, \ldots, 0), \ x = (\cos (-2\varphi_0) - 1, 0, \ldots, 0, \sin (-2\varphi_0)) \} = \{ e, d \} \tag{2.18}$$

and therefore it sends any unit circle passing through $e'$ and $d'$ to a unit circle passing through $e$ and $d$. 
Therefore, by comparing Figure 2.4 with Figure 2.1 and considering (2.4), all the unit circles passing through e and d can be parameterized as

\[ \{ x = My \mid y = (\cos \varphi + \cos \varphi_0) y' + (0, \ldots, 0, \sin \varphi), \ y' \in Y, \ 0 \leq \varphi < 2\pi \} \]  \hspace{1cm} (2.19)

With the help of this parameterization, we are ready to show the following lemma.

**Lemma 2.2.** For \( d \in C_1, d \not\in \{ e, \tilde{e} \} \), any unit circle in \( \mathbb{R}^n \), which passes through \( d \) and \( e \), has some point whose \( x_1 \)-coordinate is positive, except the circle \( C_1 \).

**Proof.** Without loss of generality, we can assume the \( x_n \)-coordinate of \( d \) is negative. Note that with \( \varphi = \pi \mp \varphi_0 \) in (2.19), \( y = (0, \ldots, 0, \pm \sin \varphi_0) \) are the points \( e' \) or \( d' \) in the \( y \)-coordinate and further \( \varphi = \pi \mp \varphi_0 \) means that

\[ x = (0, \ldots, 0) = e, \quad x = (\cos (-2\varphi_0) - 1, 0, \ldots, 0, \sin (-2\varphi_0)) = d \]  \hspace{1cm} (2.20)

in the \( x \)-coordinate, regardless of \( y' \in Y \). Any unit circle passing through \( e \) and \( d \) is given as \( x = My \) with \( y \) given as in (2.19), that is,

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_{n-1} \\
  x_n
\end{bmatrix} =
\begin{bmatrix}
  \cos \varphi_0 & 0 & \cdots & 0 & \sin \varphi_0 \\
  0 & 1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & 0 \\
  -\sin \varphi_0 & 0 & \cdots & 0 & \cos \varphi_0
\end{bmatrix}
\begin{bmatrix}
  (\cos \varphi + \cos \varphi_0) y'_1 + \cos \varphi_0 \\
  (\cos \varphi + \cos \varphi_0) y'_2 \\
  \vdots \\
  (\cos \varphi + \cos \varphi_0) y'_{n-1} \\
  \sin \varphi
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]  \hspace{1cm} (2.21)

The first coordinate is

\[ x_1 = \cos \varphi_0 (\cos \varphi + \cos \varphi_0) y'_1 + \cos^2 \varphi_0 + \sin \varphi_0 \sin \varphi - 1. \]  \hspace{1cm} (2.22)

We show that for \( y'_1 \neq -1 \) (\( y'_1 = -1 \) means the circle \( C'_1 \) in the \( y \)-coordinate and the circle \( C_1 \) in the \( x \)-coordinate, see Figure 2.4), there is always some \( \varphi \) near \( \pi - \varphi_0 \) (i.e., near the point \( e \)) such that the above \( x_1 \) becomes positive.

Let

\[ \theta = (\pi - \varphi_0) - \varphi = \pi - (\varphi + \varphi_0), \]  \hspace{1cm} (2.23)

and so

\[ \varphi = \pi - (\theta + \varphi_0). \]  \hspace{1cm} (2.24)
Then, the above is
\[ x_1 = - \cos \varphi_0 \cos (\theta + \varphi_0) y'_1 + \cos^2 \varphi_0 (1 + y'_1) + \sin \varphi_0 \sin (\theta + \varphi_0) - 1 \]
\[ = - \cos \varphi_0 [ \cos \theta \cos \varphi_0 - \sin \theta \sin \varphi_0 ] y'_1 + \sin \varphi_0 [ \sin \theta \cos \varphi_0 + \cos \theta \sin \varphi_0 ] \]
\[ - 1 + \cos^2 \varphi_0 (1 + y'_1) \]
\[ = \sin \theta \sin \varphi_0 \cos \varphi_0 (1 + y'_1) + \cos \theta \sin^2 \varphi_0 - \cos \theta \cos^2 \varphi_0 y'_1 \]
\[ - 1 + \cos^2 \varphi_0 (1 + y'_1) \]
\[ = \sin \theta \sin \varphi_0 \cos \varphi_0 (1 + y'_1) + \cos \theta - \cos \theta \cos^2 \varphi_0 (1 + y'_1) \]
\[ - [1 - \cos^2 \varphi_0 (1 + y'_1)] \]
\[ = \sin \theta \sin \varphi_0 \cos \varphi_0 (1 + y'_1) - [1 - \cos^2 \varphi_0 (1 + y'_1)] (1 - \cos \theta). \]

\( \theta = 0 (\varphi = \pi - \varphi_0) \) means the intersection point e and the above \( x_1 \) becomes 0 as it should. Assume
\[ \theta \neq 0 \quad (- \pi - \varphi_0 < \theta < 0, \ 0 < \theta \leq \pi - \varphi_0). \] (2.26)

Then, \( x_1 \) is positive if and only if
\[ \sin \theta \sin \varphi_0 \cos \varphi_0 (1 + y'_1) > [1 - \cos^2 \varphi_0 (1 + y'_1)] (1 - \cos \theta), \] (2.27)
that is,
\[ \frac{\sin \theta}{1 - \cos \theta} > \frac{1 - \cos^2 \varphi_0 (1 + y'_1)}{\sin \varphi_0 \cos \varphi_0 (1 + y'_1)} \] (2.28)
(recall \( y'_1 \neq -1 \) and \( 0 < \varphi_0 < \pi / 2 \)). In other words, the \( x_1 \)-coordinate is positive if and only if
\[ \cot \frac{\theta}{2} > \frac{1 - \cos^2 \varphi_0 (1 + y'_1)}{\sin \varphi_0 \cos \varphi_0 (1 + y'_1)}. \] (2.29)

Therefore, for \( y'_1 \neq -1 \) (i.e., except the circle \( C_1 \)), the \( x_1 \)-coordinate is positive for small enough \( \theta > 0 \).

3. Main theorem. In the previous section, we introduced all preliminary lemmas for the main result of this paper. Now, we prove our main theorem.

**Theorem 3.1.** If a one-to-one mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) maps every unit circle onto a unit circle, then \( f \) is a linear isometry up to translation.

**Proof.** We show \( f \) preserves the distance 2. Suppose the distance between \( a = f(A) \) and \( b = f(B) \) is less than 2, while the distance between \( A \) and \( B \) is 2—see Figure 3.1. Then, we show it leads to a contradiction.

Let the distance between \( a \) and \( b \) be \( 2c \ (0 < c < 1) \). Choose any unit circle \( C \) passing through \( A \) and \( B \) and let \( f(C) = C_1 \). Choose a coordinate for \( a \) and \( b \) as in Figure 3.1 such that \( C_1 \) lies in the \( x_1 \)-plane and
\[ a = (-1 - \sqrt{1 - c^2}, 0, \ldots, 0, c), \quad b = (-1 - \sqrt{1 - c^2}, 0, \ldots, 0, -c). \] (3.1)
Let
\[ e = (0, \ldots, 0), \quad \tilde{e} = (-2, 0, \ldots, 0). \] (3.2)

Let \( f(E) = e \) and \( \tilde{E} \) the antipodal point (in \( C \)) of \( E \) and let \( f(\tilde{E}) = d \). Let the union of all the unit circles passing through \( a \) and \( b \) be \( T \) and the \((n-1)\)-dimensional unit sphere passing through \( A \) and \( B \) be \( S \) and the \((n-1)\)-dimensional unit sphere passing through \( e \) and \( \tilde{e} \) be \( S_1 \).

Then, it is clear that any point \( P \) on \( S \) (\( P \not\in \{A, B\} \)) lies in some unit circle determined by the three points \( A \), \( B \), and \( P \). To see this, if we call \( O \) the common center of \( C \) and \( S \), and let
\[ \langle \overrightarrow{OP}, \overrightarrow{OA} \rangle = \sin \varphi_0 \quad \left( -\frac{\pi}{2} < \varphi_0 < \frac{\pi}{2} \right), \] (3.3)
then the unit circle determined by these three points is parameterized as
\[ \overrightarrow{OV} (\varphi) = \cos \varphi \left( \frac{\overrightarrow{OP} - \sin \varphi_0 \overrightarrow{OA}}{\cos \varphi_0} \right) + \sin \varphi \overrightarrow{OA} \quad (-\pi < \varphi \leq \pi). \] (3.4)

Note that
\[ \left\{ \left( \frac{\overrightarrow{OP} - \sin \varphi_0 \overrightarrow{OA}}{\cos \varphi_0} \right), \overrightarrow{OA} \right\}, \] (3.5)
are orthonormal to each other and
\[ \overrightarrow{OV} (\varphi_0) = \overrightarrow{OP}, \quad \overrightarrow{OV} \left( \frac{\pi}{2} \right) = \overrightarrow{OA}, \]
\[ \overrightarrow{OV} \left( -\frac{\pi}{2} \right) = -\overrightarrow{OA} = \overrightarrow{OB}. \] (3.6)

Since the image of this unit circle lies in \( T \), it follows that the image of the whole \( S \) under \( f \) lies in \( T \).
It is also obvious that the $x_1$-coordinate of any point in $T$ is nonpositive. (Note that the center of any unit circle passing through $a$ and $b$ has coordinate
\[ \sqrt{1-c^2}y - \left(1 + \sqrt{1-c^2},0,\ldots,0\right) \quad \text{for some } y \in Y, \] (3.7)
(see (2.4)) and the distance between this center and any $x = (x_1,\ldots,x_n)$ is
\[ \sqrt{\left(x_1 + 1 + \sqrt{1-c^2}(1-y_1)\right)^2 + \cdots} \] (3.8)
and because
\[ \sqrt{1-c^2}(1-y_1) \geq 0, \] (3.9)
positive $x_1$ makes the distance larger than 1, which means that if $x_1 > 0$, we have $x \notin T$.)

Now, if $d = \tilde{e}$, then the image of any unit circle passing through $E$ and $\tilde{E}$ lies in both $T$ and $S_1$. However, by Lemma 2.1, $T \cap S_1 = C_1$ and this fact contradicts the injectivity of $f$.

On the other hand, if $d \neq \tilde{e}$, the image of any unit circle, except the circle $C$, passing through $E$ and $\tilde{E}$ is a unit circle passing through $e$ and $d$. This unit circle is not $C_1$ since $f$ is one-to-one, and by Lemma 2.2 it cannot stay completely in $T$, a contradiction.

Consequently, $f$ preserves the distance. According to the well-known theorem of Beckman and Quarles, $f$ is a linear isometry up to translation.\hfill \Box

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