ON STARLIKENESS AND CLOSE-TO-CONVEXITY
OF CERTAIN ANALYTIC FUNCTIONS

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Our purpose is to derive some sufficient conditions for starlikeness and close-to-convexity of order \( \alpha \) of certain analytic functions in the open unit disk.

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1. Introduction. Let \( A_n \) be the class of functions of the form

\[
f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\})
\]  

(1.1)

which are analytic in the open unit disk \( U = \{z : |z| < 1\} \). A function \( f \in A_n \) is said to be in the class \( S_n^* (\alpha) \) if it satisfies

\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U)
\]

(1.2)

for some \( \alpha \) \((0 \leq \alpha < 1)\). A function in the class \( S_n^* (\alpha) \) is starlike of order \( \alpha \) in \( U \). We also write \( A_1 = A \) and \( S_1^* (\alpha) = S^* (\alpha) \).

Let \( C_n(\alpha) \) be the subclass of \( A_n \) consisting of functions \( f(z) \) which satisfy

\[
\text{Re}\left\{ f'(z) \right\} > \alpha \quad (z \in U)
\]

(1.3)

for some \( \alpha \) \((0 \leq \alpha < 1)\). A function \( f(z) \) in \( C_n(\alpha) \) is close-to-convex of order \( \alpha \) in \( U \) (cf. Duren [1]).

Let \( f(z) \) and \( g(z) \) be analytic in \( U \). Then the function \( f(z) \) is said to be subordinate to \( g \), written \( f \prec g \) or \( f(z) \prec g(z) \), if there exists an analytic function \( w(z) \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in U) \) such that \( f(z) = g(w(z)) \) for \( z \in U \). If \( g(z) \) is univalent in \( U \), then \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

Let \( H(p(z), zp'(z)) \prec h(z) \) be a first-order differential subordination. Then a univalent function \( q(z) \) is called its dominant if \( p(z) \prec q(z) \) for all analytic functions \( p(z) \) that satisfy the differential subordination. A dominant \( \tilde{q}(z) \) is called the best dominant if \( \tilde{q}(z) \prec q(z) \) for all dominants \( q(z) \). For the general theory of first-order differential subordination and its applications, we refer to [3].

Recently, Xu and Yang [5] obtained some results on starlikeness and close-to-convexity of certain meromorphic functions. In the present note, we investigate some
sufficient conditions for starlikeness and close-to-convexity of order \( \alpha \) of certain analytic functions in \( U \) by using the subordination principle, and obtain some useful corollaries as special cases. Furthermore, we extend the results given by Owa et al. [4].

2. Main results. To derive our results, we need the following lemmas.

**Lemma 2.1** [6]. Let \( g(z) = b_0 + b_n z^n + b_{n+1} z^{n+1} + \cdots \) \((n \in \mathbb{N})\) be analytic in \( U \) and let \( h(z) \) be analytic and starlike (with respect to the origin), univalent in \( U \) with \( h(0) = 0 \). If \( zg'(z) < h(z) \) \((z \in U)\), then

\[
g(z) < b_0 + \frac{1}{n} \int_0^z \frac{h(t)}{t} dt. \tag{2.1}
\]

**Lemma 2.2** [3]. Let \( g(z) \) be analytic and univalent in \( U \) and let \( \theta(w) \) and \( \varphi(w) \) be analytic in a domain \( D \) containing \( g(U) \), with \( \varphi(w) \neq 0 \) when \( w \in g(U) \). Set

\[
Q(z) = zg'(z) \varphi(g(z)), \quad h(z) = \theta(g(z)) + Q(z) \tag{2.2}
\]

and suppose that

(i) \( Q(z) \) is univalent and starlike in \( U \);
(ii) \( \text{Re}\{zh'(z)/Q(z)\} = \text{Re}\{\theta'(g(z))/\varphi(g(z)) + zQ'(z)/Q(z)\} > 0 \) \((z \in U)\).

If \( p(z) \) is analytic in \( U \), with \( p(0) = g(0) \), \( p(U) \subset D \), and

\[
\theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z), \tag{2.3}
\]

then \( p(z) < g(z) \) and \( g(z) \) is the best dominant of (2.3).

**Lemma 2.3** [2]. Let \( g(z) = b_0 + b_n z^n + b_{n+1} z^{n+1} + \cdots \) \((n \in \mathbb{N})\) be analytic in \( U \) with \( g(z) \neq b_0 \). If \( 0 < |z_0| < 1 \) and \( \text{Re}\{g(z_0)\} = \min_{z|z| < |z_0|} \text{Re}\{g(z)\} \), then

\[
z_0g'(z_0) \leq -\frac{n|b_0 - g(z_0)|^2}{2\text{Re}\{b_0 - g(z_0)\}}. \tag{2.4}
\]

Applying Lemma 2.1, we now derive the following.

**Theorem 2.4.** Let \( f \in A_n \) satisfy \( f(z)f''(z) \neq 0 \) for \( z \in U \setminus \{0\} \) and

\[
-\alpha \frac{zf''(z)}{f'(z)} + \frac{zf'''(z)}{f'(z)} + \alpha < \frac{az}{1-bz} \quad (z \in U), \tag{2.5}
\]

where \( \alpha, a, \) and \( b \) are real numbers with \( a \neq 0 \) and \( b \leq 1 \).

(i) If \( 0 < a \leq n \) and \( 0 < b \leq 1 \), then

\[
\text{Re}\left\{\frac{z^\alpha f'(z)}{f'(z)}\right\} > \left(\frac{1}{1+b}\right)^{a/nb} \quad (z \in U). \tag{2.6}
\]

(ii) If \( 0 < a \leq n \) and \( b = 0 \), then

\[
\text{Re}\left\{\frac{z^\alpha f'(z)}{f'(z)}\right\} > e^{-a/n} \quad (z \in U). \tag{2.7}
\]
(iii) If \( a \neq 0 \) and \( 0 < b \leq 1 \), then
\[
\left| \left( \frac{z^\alpha f'(z)}{f^\alpha(z)} \right)^{-nb/a} - 1 \right| < b \quad (z \in U).
\] (2.8)

(iv) If \( a > 0 \) and \( b = 0 \), then
\[
\left| \frac{z^\alpha f''(z)}{f^\alpha(z)} - 1 \right| < e^{a/n} - 1 \quad (z \in U).
\] (2.9)

**Proof.** Let \( f \in A_n \) with \( f(z)f'(z) \neq 0 \) for \( z \in U \setminus \{0\} \) and define
\[
g(z) = -\alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) + \frac{zf''(z)}{f'(z)}.
\] (2.10)

Then \( g(z) = b_n z^n + b_{n+1} z^{n+1} + \cdots \) is analytic in \( U \) and (2.5) can be rewritten as
\[
g(z) < h(z),
\] (2.11)

where \( h(z) = az/(1-bz) \) is analytic and starlike in \( U \). Applying Lemma 2.1 to (2.11), we have
\[
\int_0^z \frac{g(t)}{t} dt < \frac{1}{n} \int_0^z \frac{h(t)}{t} dt,
\] (2.12)

that is,
\[
-\alpha \int_0^z \left( \frac{f'(t)}{f(t)} - \frac{1}{t} \right) dt + \int_0^z \frac{f''(t)}{f'(t)} dt < \frac{a}{n} \int_0^z \frac{dt}{1-bt}. \tag{2.13}
\]

(i) If \( 0 < a \leq n \) and \( 0 < b \leq 1 \), then from (2.13) we deduce that
\[
\frac{z^\alpha f'(z)}{f^\alpha(z)} < \left( \frac{1}{1-bz} \right)^{a/nb} \equiv h_1(z).
\] (2.14)

The function \( h_1(z) \) is analytic and convex univalent in \( U \) because
\[
\text{Re} \left\{ 1 + \frac{zh_1''(z)}{h_1(z)} \right\} = \text{Re} \left\{ 1 + \frac{(a/n)z}{1-bz} \right\} \geq \frac{1-a/n}{1+b} > 0 \quad (z \in U).
\] (2.15)

Also, \( h_1(U) \) is symmetric with respect to the real axis. Hence \( \text{Re} \{h_1(z)\} > h_1(-1) \) in \( U \) and it follows from (2.14) that
\[
\text{Re} \left\{ \frac{z^\alpha f''(z)}{f^\alpha(z)} \right\} > \left( \frac{1}{1+b} \right)^{a/nb} \quad (z \in U). \tag{2.16}
\]

(ii) If \( 0 < a \leq n \) and \( b = 0 \), then from (2.13) we obtain
\[
\frac{z^\alpha f'(z)}{f^\alpha(z)} < e^{(a/n)z} \equiv h_2(z).
\] (2.17)
Since \( h_2(z) \) is analytic and convex univalent in \( U \) and \( h_2(U) \) is symmetric with respect to the real axis, it follows from (2.17) that
\[
\Re \left\{ \frac{z^\alpha f'(z)}{f^\alpha(z)} \right\} > e^{-a/n} \quad (z \in U). \tag{2.18}
\]

(iii) If \( a \neq 0 \) and \( 0 < b \leq 1 \), then by (2.14) we have
\[
\frac{z^\alpha f'(z)}{f^\alpha(z)} = \left( \frac{1}{1 - bw(z)} \right)^{a/nb} (z \in U), \tag{2.19}
\]
where \( w(z) \) is analytic in \( U \) with \( |w(z)| \leq |z| \) \((z \in U)\). Therefore we have
\[
\left| \left( \frac{z^\alpha f'(z)}{f^\alpha(z)} \right)^{-nb/a} - 1 \right| < | -bw(z) | < b \quad (z \in U). \tag{2.20}
\]

(iv) If \( a > 0 \) and \( b = 0 \), then from (2.17) we get
\[
\frac{z^\alpha f'(z)}{f^\alpha(z)} = e^{(a/n)w(z)} \quad (z \in U), \tag{2.21}
\]
where \( w(z) \) is analytic in \( U \) with \( |w(z)| \leq |z| \) \((z \in U)\). Thus
\[
\left| \frac{z^\alpha f'(z)}{f^\alpha(z)} - 1 \right| = | e^{(a/n)w(z)} - 1 | \leq e^{(a/n)|w(z)|} - 1 < e^{a/n} - 1 \quad (z \in U). \tag{2.22}
\]

Therefore the proof of Theorem 2.4 is completed. \( \Box \)

By specifying the values of the parameters appearing in Theorem 2.4, we can obtain several useful corollaries.

Taking \( 0 < a = 2(\alpha - \beta) \leq n \) and \( b = 1 \), Theorem 2.4(i) reduces to the following.

**Corollary 2.5.** Let \( f \in A_n \) satisfy \( f(z)f'(z) \neq 0 \) for \( z \in U \setminus \{ 0 \} \) and
\[
\Re \left\{ \frac{zf''(z)}{f'(z)} - \frac{zf''(z)}{f'(z)} \right\} < 2\alpha - \beta \quad (z \in U), \tag{2.23}
\]
where \( \alpha \) is a real number and \( \alpha - n/2 \leq \beta < \alpha \), then
\[
\Re \left\{ \frac{z^\alpha f'(z)}{f^\alpha(z)} \right\} > \frac{1}{2^{2(\alpha - \beta)/n}} \quad (z \in U). \tag{2.24}
\]

**Remark 2.6.** Owa et al. [4] proved that if \( f \in A_n \) satisfies \( f(z)f'(z) \neq 0 \) for \( z \in U \setminus \{ 0 \} \) and (2.23) for \( \alpha \geq 0 \) and \( \alpha - n/2 \leq \beta < \alpha \), then
\[
\Re \left\{ \frac{z^\alpha f'(z)}{f^\alpha(z)} \right\} > \frac{n}{n + 2\alpha - 2\beta} \quad (z \in U). \tag{2.25}
\]

In view of \( 2^x < 1 + x \) \((0 < x < 1)\), Corollary 2.5 is better than the main theorem of [4].

**Corollary 2.7.** If \( f \in A_n \) satisfies \( f(z)f'(z) \neq 0 \) for \( z \in U \setminus \{ 0 \} \) and
\[
\Re \left\{ \frac{zf''(z)}{f'(z)} - \frac{zf''(z)}{f'(z)} \right\} < 1 + \frac{a}{2} \quad (z \in U) \tag{2.26}
\]
for some \( a \) \((0 < a \leq n)\), then \( f \in S_n^*(2^{-a/n}) \) and the order \( 2^{-a/n} \) is sharp.
PROOF. Letting $\alpha = b = 1$ in Theorem 2.4(i) and using (2.26), we see that $f \in S_n^*(2^{-a/n})$. To show that the order $2^{-a/n}$ cannot be increased, we consider

$$f(z) = \exp \int_0^z \frac{(1+t^n)^{-a/n}}{t} \, dt \in A_n.$$  

(2.27)

It is easy to verify that the function $f(z)$ defined by (2.27) satisfies (2.26) and

$$\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} = \Re \left\{ \left( \frac{1}{1+z^n} \right)^{a/n} \right\} - \left( \frac{1}{2} \right)^{a/n}$$  

(2.28)

as $z \to 1$. Therefore the proof is completed.

Putting $\alpha = 0$ and $b = 1$ in Theorem 2.4(i), we have the following.

**Corollary 2.8.** If $f \in A_n$ satisfies $f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and

$$-\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} < \frac{a}{2}$$  

(2.29)

for some $a$ ($0 < a \leq n$), then $f \in C_n(2^{-a/n})$ and the order $2^{-a/n}$ is sharp.

**Remark 2.9.** Corollary 2.7 (with $0 < a = 2(1-\beta) \leq n$) and Corollary 2.8 (with $0 < a = 2\beta < n$) are better than the corresponding results in [4].

Setting $\alpha = 0$ and 1 in Theorem 2.4(ii), we have the following two corollaries.

**Corollary 2.10.** If $f \in A_n$ satisfies $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and

$$\left| \frac{zf''(z)}{f'(z)} \right| < a$$  

(2.30)

for some $a$ ($0 < a \leq n$), then $f \in C_n(e^{-a/n})$.

**Corollary 2.11.** If $f \in A_n$ satisfies $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < a$$  

(2.31)

for some $a$ ($0 < a \leq n$), then $f \in S_n^*(e^{-a/n})$ and the order $e^{-a/n}$ is sharp with the extremal function

$$f(z) = \exp \int_0^z \frac{e^{-(a/n)t^n}}{t} \, dt.$$  

(2.32)

For $\alpha = 1$ and $a = -nb$ ($0 < b \leq 1$) in Theorem 2.4(iii), we have the following.

**Corollary 2.12.** If $f \in A_n$ satisfies $f(z)f'(z) \neq 0$ for $z \in U \setminus \{0\}$ and

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} < -\frac{-nbz}{1-bz}$$  

(2.33)

for some $b$ ($0 < b \leq n$), then $f \in S_n^*(1-b)$ and the order $1-b$ is sharp with the extremal function $f(z) = ze^{(b/n)z^n}$. 


Next, applying Lemma 2.2, we obtain the following two results.

**Theorem 2.13.** Let \( f \in A \) satisfy \( f(z) \neq 0 \) for \( z \in U \setminus \{0\} \) and

\[
\frac{zf'(z)}{f(z)} + \frac{z^2f''(z)}{f(z)} < h(z) \quad (z \in U),
\]

where

\[
h(z) = \frac{(1-2\alpha)^2z^2 + 2(2-3\alpha) + 1}{(1-z)^2} \quad (0 \leq \alpha < 1; \ z \in U),
\]

then \( f \in S^*(\alpha) \) and the order \( \alpha \) is sharp.

**Proof.** We put

\[
\frac{zf'(z)}{f(z)} = (1-\alpha)p(z) + \alpha
\]

for \( 0 \leq \alpha < 1 \). Then \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Differentiating (2.36) logarithmically, we find that

\[
\frac{zf'(z)}{f(z)} + \frac{z^2f''(z)}{f(z)} = (1-\alpha)zp'(z) + ((1-\alpha)p(z) + \alpha)^2.
\]

From (2.34) and (2.37), we have

\[
(1-\alpha)zp'(z) + (1-\alpha)^2p^2(z) + 2\alpha(1-\alpha)p(z) + \alpha^2 < h(z).
\]

Now we choose

\[
g(z) = \frac{1+z}{1-z}, \quad \theta(w) = (1-\alpha)^2w^2 + 2(1-\alpha)w + \alpha^2, \quad \varphi(w) = 1-\alpha.
\]

Then \( g(z) \) is analytic and univalent in \( U \), \( \text{Re}\{g(z)\} > 0 \ (z \in U) \), and \( \theta(w) \) and \( \varphi(w) \) are analytic with \( \varphi(w) \neq 0 \) in the \( w \)-plane.

The function

\[
Q(z) = zg'(z)\varphi(z) = 2(1-\alpha)\frac{z}{(1-z)^2}
\]

is univalent and starlike in \( U \). Further,

\[
\theta(g(z)) + Q(z) = (1-\alpha)^2\left(\frac{1+z}{1-z}\right)^2 + 2\alpha(1-\alpha)\left(\frac{1+z}{1-z}\right) + \alpha^2 + 2(1-\alpha)\frac{z}{1-z}
\]

\[
= \frac{(1-2\alpha)^2z^2 + 2(2-3\alpha)z + 1}{(1-z)^2} = h(z),
\]

\[
\text{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \text{Re}\left\{2(1-\alpha)g(z) + 2\alpha + \frac{zQ'(z)}{Q(z)}\right\}
\]

\[
= (3-2\alpha)\text{Re}\left\{\frac{1+z}{1-z}\right\} + 2\alpha > 0
\]
for \( z \in U \). In view of (2.38)-(2.42), we see that
\[
\theta(p(z)) + z p'(z) \varphi(p(z)) < \theta(g(z)) + z g'(z) \varphi(g(z)) = h(z).
\]
(2.43)

Therefore, Lemma 2.2 leads to \( p(z) < g(z) \), which implies that \( f \in S^*(\alpha) \). Next, we consider
\[
f(z) = \frac{z}{(1-z)^{1-\alpha}} \in A.
\]
(2.44)

It is easy to see that
\[
\frac{zf'(z)}{f(z)} + 2 \alpha \frac{z^2 f''(z)}{f(z)} = h(z),
\]
\[
\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} = \frac{1 + (1 - 2 \alpha)z}{1-z} \quad \rightarrow \quad \alpha
\]
as \( z \rightarrow -1 \). The proof of the theorem is completed.

**Theorem 2.14.** If \( f \in A \) satisfies \( f(z) \neq 0 \) for \( z \in U \setminus \{0\} \) and
\[
\frac{zf'(z)}{f(z)} + 2 \alpha \frac{z^2 f''(z)}{f(z)} < h(z),
\]
(2.46)

where
\[
h(z) = \frac{(2 \alpha - 1)^3 z^2 + 2 \alpha (3 - 4 \alpha) z + 1}{(1-z)^2} \quad (0 \leq \alpha < 1; \ z \in U),
\]
(2.47)

then \( f \in S^*(\alpha) \) and the order \( \alpha \) is sharp.

**Proof.** It suffices to prove the theorem for \( 0 < \alpha < 1 \). We define the function \( p(z) \) by (2.36). Then \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). By a simple calculation, we find that
\[
\frac{zf'(z)}{f(z)} + 2 \alpha \frac{z^2 f''(z)}{f(z)} = 2 \alpha (1-\alpha) z p'(z) + 2 \alpha (1-\alpha)^2 p^2(z) + (1-\alpha) (1-2 \alpha + 4 \alpha^2) p(z)
\]
\[
+ \alpha (1-2 \alpha + 2 \alpha^2).
\]
(2.48)

Thus the subordination (2.46) becomes
\[
2 \alpha (1-\alpha) z p'(z) + 2 \alpha (1-\alpha)^2 p^2(z) + (1-\alpha) (1-2 \alpha + 4 \alpha^2) p(z)
\]
\[
+ \alpha (1-2 \alpha + 2 \alpha^2) < h(z).
\]
(2.49)

Set \( g(z) = (1+z)/(1-z) \), \( \theta(w) = 2 \alpha (1-\alpha)^2 w^2 + (1-\alpha) (1-2 \alpha + 4 \alpha^2) w + (1-2 \alpha + 2 \alpha^2) \), and \( \varphi(w) = 2 \alpha (1-\alpha) \). Then \( g(z) \), \( \theta(w) \), and \( \varphi(w) \) satisfy the conditions of Lemma 2.2. The function
\[
Q(z) = z g'(z) \varphi(g(z)) = 4 \alpha (1-\alpha) \frac{z}{(1-z)^2}
\]
(2.50)
is univalent and starlike in $U$. Further,

$$\theta(g(z)) + Q(z) = 2\alpha(1-\alpha)^2 \left(\frac{1+z}{1-z}\right)^2 + (1-\alpha)(1-2\alpha + 4\alpha^2) \left(\frac{1+z}{1-z}\right)$$

$$+ \alpha(1-2\alpha+2\alpha^2) + 4\alpha(1-\alpha) \frac{z}{(1-z)^2}$$

$$= \frac{(2\alpha-1)^3 z^2 + 2\alpha(3-4\alpha)z + 1}{(1-z)^2} = h(z), \quad (2.51)$$

for $z \in U$. Note that

$$\theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z). \quad (2.52)$$

Hence, an application of Lemma 2.2 yields that $p(z) < g(z)$, that is, $f \in S^*(\alpha)$. For the function $f(z)$ defined by (2.44), we have

$$\frac{zf'(z)}{f(z)} + 2\alpha z^2 f'''(z) = h(z), \quad (2.53)$$

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \to \alpha \quad \text{as} \quad z \to -1.$$

Therefore we complete the proof of Theorem 2.14. \qed

Finally, by using Lemma 2.3, we prove the following.

**Theorem 2.15.** Let $f \in A_n$ satisfy $f(z) \neq 0$ for $z \in U \setminus \{0\}$ and

$$\left| \arg \left\{ (1-\lambda) \frac{z^2 (f'(z))^2}{f^2(z)} + \lambda \left( \frac{zf'(z)}{f(z)} + \frac{z^2 f'''(z)}{f(z)} \right) + \frac{n\lambda}{2} \right\} \right| < \pi \quad (z \in U) \quad (2.54)$$

for some $\lambda$ ($\lambda > 0$). Then $f \in S_n^*(0)$ and the order 0 is sharp.

**Proof.** The function $g(z)$ defined by

$$g(z) = \frac{zf'(z)}{f(z)} = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots \quad (2.55)$$

is analytic in $U$ and it is easily verified that

$$(1-\lambda) \frac{z^2 (f'(z))^2}{f^2(z)} + \lambda \left( \frac{zf'(z)}{f(z)} + \frac{z^2 f'''(z)}{f(z)} \right) = g^2(z) + \lambda zg'(z) \quad (\lambda > 0; \ z \in U). \quad (2.56)$$

Suppose that there exists a point $z_0 \in U \setminus \{0\}$ such that

$$\text{Re} \{g(z)\} > 0 \quad (|z| < |z_0|), \quad g(z_0) = i\beta, \quad (2.57)$$
where $\beta$ is a real number. Then, applying Lemma 2.3, we have

$$z_0 g'(z_0) \leq -\frac{n(1+\beta^2)}{2}. \quad (2.58)$$

Thus it follows from (2.56), (2.57), and (2.58) that

$$\begin{align*}
(1-\lambda)\frac{z_0^2 (f'(z_0))^2}{f(z_0)} + \lambda \left( \frac{zf'(z_0)}{f(z_0)} + \frac{z_0^2 f''(z_0)}{f(z_0)} \right) + \frac{n\lambda}{2} \\
= (g(z_0))^2 + \lambda z_0 g'(z_0) + \frac{n\lambda}{2}
\end{align*} \quad (2.59)$$

$$\leq -\beta^2 - \frac{n\lambda(1+\beta^2)}{2} + \frac{n\lambda}{2} \leq 0$$

for $\lambda > 0$, which contradicts (2.54). Hence $\text{Re} \{g(z)\} > 0 \ (z \in U)$, that is $f \in S^{n^*}_n(0)$. If we let

$$f_n(z) = \frac{z}{(1-z^n)^{1/n}} \in A_n, \quad (2.60)$$

then

$$\begin{align*}
(1-\lambda)\frac{f_n'(z)^2}{f_n(z)} + \lambda \left( \frac{zf_n'(z)}{f_n(z)} + \frac{z^2 f_n''(z)}{f_n(z)} \right) + \frac{n\lambda}{2} \\
= \left(1 + \frac{n\lambda}{2}\right) \left( \frac{1+z^n}{1-z^n} \right)^2 \quad (z \in U),
\end{align*} \quad (2.61)$$

and so the function $f_n(z)$ satisfies (2.54). Noting that

$$\text{Re} \frac{z f_n'(z)}{f_n(z)} = \text{Re} \frac{1+z^n}{1-z^n} \to 0 \quad (2.62)$$
as $z \to e^{i\pi/n}$, we conclude that the order 0 is the best possible.

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**References**


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