ON THE GENUS OF FREE LOOP FIBRATIONS OVER $F_0$-SPACES

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We give a lower bound of the genus of the fibration of free loops on an elliptic space whose rational cohomology is concentrated in even degrees.


1. Introduction. In this note, all spaces are supposed to be connected and having the rational homotopy type of a CW complex of finite type. The LS category, $\text{cat}(X)$, of a space $X$ is the least integer $n$ such that $X$ can be covered by $n+1$ open subsets, each contractible in $X$. The genus, $\text{genus}(\eta)$ or $\text{genus}(p)$, of a fibration $\eta: F \to E \to B$ is the least integer $n$ such that $B$ can be covered by $n+1$ open subsets, over each of which $p$ is a trivial fibration, in the sense of fiber homotopy type. The sectional category, $\text{secat}(\eta)$, is the least integer $n$ such that $B$ can be covered by $n+1$ open subsets, over each of which $p$ has a section. Let

\[ \mathcal{L}_X: \Omega X \to LX \to X \]  

be the fibration of free loops on a 2-connected space $X$ and let $\mathcal{P}_X: \Omega X \to PX \to X$ be the path fibration. It is known that $\mathcal{L}_X$ is an interesting object in topology and geometry [1, 9]. We know that $\text{cat}(X) = \text{secat}(\mathcal{P}_X) = \text{genus}(\mathcal{P}_X)$ (see [4, page 599]). On the other hand, since $\mathcal{L}_X$ has a section, $\text{secat}(\mathcal{L}_X) = 0$. But it seems hard to know $\text{genus}(\mathcal{L}_X)$ in general. In this note, we consider a certain case for $X$ by using the argument of the Sullivan minimal model in [4].

A simply connected space is said to be elliptic if the dimensions of rational cohomology and homotopy are finite. An elliptic space $X$ is said to be an $F_0$-space if the rational cohomology is concentrated in even degrees. Then there is an isomorphism $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ with a regular sequence $f_1, \ldots, f_n$. For example, the homogeneous space $G/H$ where $G$ and $H$ have same rank is an $F_0$-space. Note that there is a conjecture of Halperin for an $F_0$-space (see [3, page 516], [7]).

Theorem 1.1. Let $X$ be a 2-connected $F_0$-space of $n$ variables. Then $\text{genus}(\mathcal{L}_X) \geq n$.

In the following, Section 2 is a preliminary in Sullivan minimal models and we prove the theorem in Section 3. Refer to [3] for the rational model theory.

2. Sullivan model of classifying map. Let $M(X) = (\Lambda V, d)$ be the Sullivan minimal model [3, Section 12] of a 2-connected space $X$, in which $V = \bigoplus_{i>2} V^i$ as a graded vector space. Let $\nabla^i = V^{i+1}$ and let $\beta: \Lambda V \otimes \Lambda V \to \Lambda V \otimes \Lambda V$ be the derivation $(\beta(x \cdot y) = \beta(x) \cdot y + (-1)^{\deg x} x \beta(y))$ of degree −1 with the properties $\beta(v) = \tau$ and $\beta(\tau) = 0$. 
Then $M(\Omega X) = (\Lambda V,0)$ and $M(LX) \cong (\Lambda V \otimes \Lambda V, \delta)$ with $\delta v = dv$ and $\delta v = -\beta dv = \sum_j \partial d v / \partial v_j \cdot v_j$ for a basis $v_j$ of $V$ [9].

Let $Y$ be a simply connected space and let $\text{Der}_i M(Y)$ be the set of derivations of $M(Y)$ decreasing the degree by $i > 0$. We denote $\bigoplus_{i>0} \text{Der}_i M(Y)$ by $\text{Der} M(Y)$. The Lie bracket is defined by $[\sigma, \tau] = \sigma \circ \tau - (-1)^{\deg \sigma \deg \tau} \tau \circ \sigma$. The boundary operator $\partial : \text{Der} \, M(Y) \to \text{Der} \, M(Y)$ is defined by $\partial(\sigma) = [d, \sigma]$. Let $B \text{aut} Y$ be the Dold-Lashof classifying space [2] for fibrations with fiber $Y$ and $\tilde{B} \text{aut} Y$ the universal covering. The differential graded Lie algebra $L = (\text{Der} \, M(Y), \partial)$ is a model for $\tilde{B} \text{aut} Y$ (see [8, page 313]).

Any fibration with fiber $Y$ over a simply connected space $B$ is the pullback of the universal fibration by a classifying map $h : B \to \tilde{B} \text{aut} Y$. Let $Y \to E \to B$ be a fibration whose model [3, Section 15] is

$$M(B) = (\Lambda W, d) \to (\Lambda W \otimes \Lambda V, D) \to (\Lambda V, \delta) = M(Y).$$

(2.1)

Take a basis $a_i$ of $(\Lambda W)^\ast$, then there are derivations $\theta_i$ of $\Lambda V$ such that for each $z \in V$, we have $D(z) = \delta \theta_i (z)$. A differential graded algebra model for $\tilde{B} \text{aut} Y$ is given by the cochain algebra $C^\ast (L)$ [3, 23(a)] on $L$, and a model for the classifying map of the fibration $h$ is given by

$$h^* : C^\ast (L) = \text{Hom}(\text{Der} \, M(Y), Q) \to \Lambda W, \quad h^*(\psi) = \sum_i a_i \psi(\theta_i) \quad (2.2)$$

(see [6, Section 9]). Put the derivation which sends a generator $p$ to an element $q$ and other generators to zero as $(p,q)$ and the dual element with the degree shifted by $+1$ as $(s(p,q))^\ast$.

**Lemma 2.1.** The fibration $\mathcal{L}_X$ is the pullback of the universal fibration by a classifying map $h : X \to \tilde{B} \text{aut} \Omega X$, where the model is given by $h^*(s(\sigma_i, \sigma_j)^\ast) = \pm_{i,j} \partial d v_i / \partial v_j$ for $v_i, v_j \in V$ and $h^*(\text{other}) = 0$.

**Proof.** The category, $\text{cat}(f)$, of a map $f : X \to Y$ is the least integer $n$ such that $X$ can be covered by $n+1$ open subsets $U_i$, for which the restriction of $f$ to each $U_i$ is null-homotopic. Note that $\text{cat}(f) \leq \text{cat}(X)$. Recall that if $\eta : F \to E \to B$ is a simply connected fibration, then $\text{genus}(\eta) = \text{cat}(h)$ for the classifying map of $\eta$, $h : B \to \tilde{B} \text{aut} F$ [5].

**Proof of Theorem 1.1.** Let $M(X) = (\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_n), d)$ with $\deg x_i$ even, $\deg y_i$ odd, $d(x_i) = 0$, and $d(y_i) = f_i \neq 0 \in \Lambda(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$. Then $M(\Omega X) = (\Lambda(x_1, \ldots, x_n, \bar{y}_1, \ldots, \bar{y}_n), 0)$ with $\deg \bar{v} = \deg v - 1$ for any element $v$. The minimal model of the space $LX$ of free loops on $X$ is given by

$$M(LX) = (\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_n, \bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_n), \delta),$$

(3.1)

where $\delta x_i = \delta \bar{x}_i = 0$, $\delta y_i = dy_i = f_i$, $\delta \bar{y}_i = -\sum_{j=1}^n \partial f_i / \partial x_j \cdot \bar{x}_j$. Then we see from Lemma 2.1 that

$$h^*(s(\bar{y}_i, \bar{x}_j)^\ast) = -\frac{\partial f_i}{\partial x_j} \quad \text{for } 1 \leq i, j \leq n, \quad h^*(\text{other}) = 0.$$  

(3.2)
Let $J$ be the determinant of the matrix whose $(i,j)$-component is $s(\gamma_i, x_j)^*$. Then $(-1)^n h^*(J)$ is the Jacobian $| (\partial f_i / \partial x_j)_{1 \leq i, j \leq n} |$ of $f_1, \ldots, f_n$ and it is a cocycle which is not cohomologous to zero in $M(X)$ [7, Theorem B]. Therefore, as in [4, page 598(2)],

$$\text{genus}(\mathcal{L}_X) = \text{cat}(h) \geq \text{nil}(\text{Im} \tilde{H}(h^*)) \geq n,$$

(3.3)

where $\text{nil} R$ is the least integer $n$ such that $R^{n+1} = 0$ for a ring $R$ and $\tilde{H}(h^*)$ is the induced morphism in reduced cohomology.

**Corollary 3.1.** If $X$ is an $F_0$-space of $n$ variables with $\text{cat}(X) = n$, then $\text{genus}(\mathcal{L}_X) = n$.

**Example 3.2.** Let $X = S^{2n} \vee S^{2n} \cup e^{4n} \neq S^{2n} \vee S^{2n} \vee S^{4n}$. $X$ is an $F_0$-space where $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(x_1^2 + ax_2^2, x_1 x_2)$ with some $a \neq 0 \in \mathbb{Q}$ and $\deg x_i = 2n$. Then from Theorem 1.1 and [3, Lemma 27.3], $2 \leq \text{genus}(\mathcal{L}_X) \leq \text{cat}(X) \leq \text{cat}(S^{2n} \vee S^{2n}) + 1 = 2$, that is, $\text{genus}(\mathcal{L}_X) = \text{cat}(X) = 2$.

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**References**


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