Let $X$ and $Y$ be Banach spaces. A set $\mathcal{M}$ of $1$-summing operators from $X$ into $Y$ is said to be uniformly summing if the following holds: given a weakly $1$-summing sequence $(x_n)$ in $X$, the series $\sum_n \|Tx_n\|$ is uniformly convergent in $T \in \mathcal{M}$. We study some general properties and obtain a characterization of these sets when $\mathcal{M}$ is a set of operators defined on spaces of continuous functions.

2000 Mathematics Subject Classification: 47B38, 47B10.

1. Introduction. Throughout this paper, $X$ and $Y$ will be Banach spaces. If $X$ is a Banach space, $B_X = \{x \in X : \|x\| \leq 1\}$ will denote its closed unit ball and $X^*$ will be the topological dual of $X$. Given a real number $p \in [1, \infty)$, a (linear) operator $T : X \to Y$ is said to be $p$-summing if there exists a constant $C > 0$ such that

$$\left( \sum_{i=1}^{n} \|Tx_i\|^p \right)^{1/p} \leq C \sup \left\{ \left( \sum_{i=1}^{n} |\langle x^*, x_i \rangle|^p \right)^{1/p} : x^* \in B_{X^*} \right\} \quad (1.1)$$

for every finite set $\{x_1, \ldots, x_n\} \subset X$. The least $C$ for which the above inequality always holds is denoted by $\pi_p(T)$ (the $p$-summing norm of $T$). The linear space of all $p$-summing operators from $X$ into $Y$ is denoted by $\Pi_p(X,Y)$ which is a Banach space endowed with the $p$-summing norm.

As usual, $\ell^p_w(X)$ will be the Banach space of weakly $p$-summable sequences in $X$, that is, the sequences $(x_n) \subset X$ satisfying $\sum_n |\langle x^*, x_n \rangle|^p < \infty$ for all $x^* \in X^*$; the norm in $\ell^p_w(X)$ is $\epsilon_p(x_n) = \sup \{(\sum_n |\langle x^*, x_n \rangle|^p)^{1/p} : x^* \in B_{X^*}\}$. The set of all strongly $p$-summable sequences in $X$ is denoted by $\ell^p_s(X)$; the norm in this space is $\pi_p(x_n) = (\sum_n \|x_n\|^p)^{1/p}$. If $T \in \Pi_p(X,Y)$, the correspondence $\hat{T} : (x_n) \mapsto (Tx_n)$ always induces a bounded operator from $\ell^p_w(X)$ into $\ell^p_s(Y)$ with $\|\hat{T}\| = \pi_p(T)$ [5, Proposition 2.1].

Families of operators arise in different applications: equations containing a parameter, homotopies of operators, and so forth. In these applications, it may be very interesting to know that, given a set $\mathcal{M} \subset \Pi_p(X,Y)$ and $(x_n) \in \ell^p_w(X)$, the series $\sum_n \|Tx_n\|^p$ is uniformly convergent in $T \in \mathcal{M}$. The main purpose of this paper is to study uniformly $p$-summing sets, that is, those sets $\mathcal{M} \subset \Pi_p(X,Y)$ for which, given $(x_n) \in \ell^p_w(X)$, the series $\sum_n \|Tx_n\|^p$ is uniformly convergent in $T \in \mathcal{M}$. These sets also enjoy some properties that justify their study; the next proposition lists some of them.
PROPOSITION 1.1. (a) Let \((T_k)\) be a sequence in \(\Pi_p(X,Y)\). Then, \(\tilde{T}_k \xrightarrow{k} 0\) pointwise if and only if \(T_k \xrightarrow{k} 0\) pointwise and \((T_k)\) is uniformly \(p\)-summing.

(b) Let \(\mathcal{M} \subset \Pi_p(X,Y)\) be a uniformly \(p\)-summing set. If \(\mathcal{M}\) is endowed with the strong operator topology, then the map \(T \in \mathcal{M} \mapsto \sum_n \|Tx_n\|^p \in \mathbb{R}\) is continuous for every \((x_n) \in \ell^p_w(X)\).

A basic argument shows that uniformly \(p\)-summing sets are bounded for the \(p\)-summing norm. In fact, if \(X\) does not contain any copy of \(c_0\), bounded sets and uniformly \(1\)-summing sets are the same. That is the reason for which we only consider operators defined on a \(\mathcal{C}(\Omega)\)-space, \(\Omega\) being a compact Hausdorff space. We recall that every weakly compact operator \(T: \mathcal{C}(\Omega) \to Y\) has a representing measure \(m_T: \Sigma \to Y\) defined by \(m_T(B) = T^{**}(\chi_B)\) for all \(B \in \Sigma\), where \(\Sigma\) denotes the Borel \(\sigma\)-field of subsets of \(\Omega\) and \(\chi_B\) denotes the characteristic function of \(B\). The vector measure \(m_T\) is regular and countably additive [6, Theorem VI.2.5 and Corollary VI.2.14]. If we denote by \(\tilde{T}\) the operator \(T^{**}\) restricted to \(B(\Sigma)\) (the space of all bounded Borel-measurable scalar-valued functions defined on \(\Omega\)), then

\[
\tilde{T} \varphi = \int_{\Omega} \varphi \, dm_T,
\]

for all \(\varphi \in B(\Sigma)\) (the integral is the elementary Bartle integral [6, Definition I.1.12]).

It is well known that every \(p\)-summing operator defined on a Banach space \(X\) is weakly compact. In Section 2, we consider \(1\)-summing operators \(T\) defined on \(\mathcal{C}(\Omega)\); these operators are characterized as those with representing measure \(m_T\) having finite variation and \(m_1(T) = |m_T|(\Omega)\) [6, Theorem VI.3.3]. We show that a set \(\mathcal{M} \subset \Pi_1(\mathcal{C}(\Omega), Y)\) is uniformly \(1\)-summing if and only if the family of all variation measures \(|m_T|: T \in \mathcal{M}\) is uniformly bounded and there is a countably additive measure \(\mu: \Sigma \to [0, \infty)\) such that \(|m_T|: T \in \mathcal{M}\) is uniformly \(\mu\)-continuous.

In Section 3, we mention a special class of uniformly \(p\)-summing operators: uniformly dominated sets. The relationship between uniformly summing sets and relatively weak compactness is also studied. Finally, we give some examples and open problems.

2. Uniformly \(1\)-summing sets in \(\Pi_1(\mathcal{C}(\Omega), Y)\). Before facing our main theorem, we include three results which correspond to the vector measure theory. These results will be usually invoked along the following lines.

PROPOSITION 2.1 [6, Proposition I.1.17]. The following statements about a collection \(\{m_i: i \in I\}\) of \(Y\)-valued measures defined on a \(\sigma\)-field \(\Sigma\) are equivalent:

(a) the set \(\{m_i: i \in I\}\) is uniformly countably additive, that is, if \((E_n)\) is a sequence of pairwise disjoint members of \(\Sigma\), then \(\lim_n \|\sum_{k \geq n} m_i(E_k)\| = 0\) uniformly in \(i \in I\),

(b) the set \(\{\gamma^{*} \circ m_i: i \in I, \gamma^{*} \in B_{Y^{*}}\}\) is uniformly countably additive,

(c) if \((E_n)\) is a sequence of pairwise disjoint members of \(\Sigma\), then \(\lim_n \|m_i(E_n)\| = 0\) uniformly in \(i \in I\),

(d) if \((E_n)\) is a sequence of pairwise disjoint members of \(\Sigma\), then \(\lim_n \|m_i\|(E_n) = 0\) uniformly in \(i \in I\), where \(\|m_i\|\) denotes the semivariation of \(m_i\),

(e) the set \(\{|\gamma^{*} \circ m_i|: i \in I, \gamma^{*} \in B_{Y^{*}}\}\) is uniformly countably additive.
Theorem 2.2 [6, Theorem I.2.4]. Let \( \{ m_i : \Sigma \to Y : i \in I \} \) be a uniformly bounded (with respect to the semivariation) family of countably additive vector measures on a \( \sigma \)-field \( \Sigma \). The family \( \{ m_i : i \in I \} \) is uniformly countably additive if and only if there exists a positive real-valued countably additive measure \( \mu \) on \( \Sigma \) such that \( \{ m_i : i \in I \} \) is uniformly \( \mu \)-continuous, that is,

\[
\lim_{\mu(E)\to 0} \| m_i(E) \| = 0
\] (2.1)

uniformly in \( i \in I \).

If \( \Omega \) is a compact Hausdorff space and \( \Sigma \) denotes the \( \sigma \)-field of the Borel subsets of \( \Omega \), a vector measure \( m \) on \( \Sigma \) is regular if for each Borel set \( E \) and \( \varepsilon > 0 \) there exists a compact set \( K \) and an open set \( O \) such that \( K \subset E \subset O \) and \( \| m \| (O \setminus K) < \varepsilon \).

Proposition 2.3 [6, Lemma VI.2.13]. Let \( \mathcal{X} \) be a family of regular (countably additive) scalar measures defined on \( \Sigma \). Each of the following statements implies all the others:

- (a) for each pairwise disjoint sequence \( (O_n) \) of open subsets of \( \Omega \), \( \lim_n \mu(O_n) = 0 \) uniformly in \( \mu \in \mathcal{X} \),
- (b) for each pairwise disjoint sequence \( (O_n) \) of open subsets of \( \Omega \), \( \lim_n |\mu|(O_n) = 0 \) uniformly in \( \mu \in \mathcal{X} \),
- (c) \( \mathcal{X} \) is uniformly countably additive,
- (d) \( \mathcal{X} \) is uniformly regular, that is, if \( E \in \Sigma \) and \( \varepsilon > 0 \), then there exists a compact set \( K \) and an open set \( O \) such that \( K \subset E \subset O \) and sup\( \mu \in \mathcal{X} \) \( |\mu|(O \setminus K) < \varepsilon \).

Now, we are able to show our main result. In the proof, we will use the fact that \( |m_T| \) is regular when \( T : C(\Omega) \to Y \) is 1-summing [7, Proposition 15.21].

Theorem 2.4. Let \( \mathcal{M} \subset \Pi_{\mathcal{C}} (C(\Omega), Y) \) be a bounded set. The following statements are equivalent:

- (a) \( \mathcal{M} \) is uniformly 1-summing,
- (b) the family of nonnegative measures \( \{ |m_T| : T \in \mathcal{M} \} \) is uniformly countably additive,
- (c) given \( \varepsilon > 0 \) and a disjoint sequence \( (E_n) \) of Borel subsets of \( \Omega \), there exists \( n_0 \in \mathbb{N} \) such that

\[
\sum_{n \geq n_0} \| m_T(E_n) \| < \varepsilon,
\] (2.2)

for all \( T \in \mathcal{M} \).

Proof. (a) \( \Rightarrow \) (b). According to [6, Lemma VI.2.13], it suffices to show that \( \lim_{n \to \infty} |m_T|(O_n) = 0 \) uniformly in \( T \in \mathcal{M} \), for all disjoint sequences \( (O_n) \) of open subsets of \( \Omega \). By contradiction, suppose that there exists \( \varepsilon > 0 \), a sequence \( (T_n) \) in \( \mathcal{M} \), and a strictly increasing sequence \( (k_n) \) of natural numbers such that

\[
|m_{T_n}|(O_{k_n}) > 2\varepsilon, \quad \forall n \in \mathbb{N}.
\] (2.3)
Now we consider the operators $S_n : \mathcal{C}(\Omega, O_{k_n}) \to Y$ defined by
\[
S_n \varphi = \int_{O_{k_n}} \varphi \, dm_{T_n}, \quad (2.4)
\]
for all $\varphi \in \mathcal{C}(\Omega, O_{k_n})$, where $\mathcal{C}(\Omega, O_{k_n})$ is the closed subspace of $\mathcal{C}(\Omega)$ formed by all continuous functions $\varphi$ on $\Omega$ such that $\varphi$ vanishes in $\Omega \setminus O_{k_n}$. It is known that $\pi_1(S_n) = |m_{T_n}|(O_{k_n})$, for all $n \in \mathbb{N}$ [7, Theorem 19.3]. For each $n \in \mathbb{N}$, we can choose a finite set $\{\varphi_1^n, \ldots, \varphi_n^n\} \subseteq \mathcal{C}(\Omega, O_{k_n})$ satisfying $\epsilon_1(\varphi_1^n) \leq 1$ and
\[
\sum_{i=1}^{p_n} ||S_n \varphi_i^n|| > \pi_1(S_n) - \epsilon. \quad (2.5)
\]
Since the open sets $O_{k_n}$ are disjoint, it follows that the sequence $(\varphi_1^n, \ldots, \varphi_1^{p_1}, \varphi_2^n, \ldots, \varphi_2^{p_2}, \ldots)$ belongs to $\ell^1_w(\mathcal{C}(\Omega))$. Nevertheless, for all $n \in \mathbb{N}$, we have
\[
\sum_{m \geq n} \sum_{i=1}^{p_m} ||T_n \varphi_i^m|| \geq \sum_{i=1}^{p_n} ||S_n \varphi_i^n|| > \pi_1(S_n) - \epsilon = |m_{T_n}|(O_{k_n}) - \epsilon > \epsilon. \quad (2.6)
\]
This denies (a) and proves that (a) implies (b).

(b)⇒(c). Again we proceed by contradiction. Suppose $(E_n)$ is a disjoint sequence of Borel subsets of $\Omega$ for which there exists $\epsilon > 0$, a sequence $(T_n)$ in $\mathcal{M}$, and a strictly increasing sequence $(k_n)$ of natural numbers so that
\[
\sum_{i=k_n}^{k_{n+1}} ||m_{T_n}(E_i)|| > \epsilon, \quad \forall n \in \mathbb{N}. \quad (2.7)
\]
If we put $B_n = \bigcup_{i=k_n+1}^{k_{n+1}} E_i$, the above inequality yields $|m_{T_n}|(B_n) > \epsilon$. So, in view of [6, Proposition I.1.17], the family $\{|m_T| : T \in \mathcal{M}\}$ is not uniformly countably additive.

(c)⇒(b). We need to prove
\[
\lim_{n \to \infty} |m_T|(E_n) = 0 \quad \text{uniformly in } T \in \mathcal{M}, \quad (2.8)
\]
for all disjoint sequences $(E_n)$ of Borel subsets of $\Omega$. Suppose (b) fails. Then, there exists $\epsilon > 0$, a sequence $(T_n)$ in $\mathcal{M}$, and a strictly increasing sequence $(k_n)$ of natural numbers satisfying
\[
|m_{T_n}|(E_{k_n}) > \epsilon, \quad \forall n \in \mathbb{N}. \quad (2.9)
\]
For each $n \in \mathbb{N}$, we choose a finite partition $\{E_1^n, \ldots, E_{p_n}^n\}$ of $E_{k_n}$ for which
\[
\sum_{i=1}^{p_n} ||m_{T_n}(E_i^n)|| > \epsilon. \quad (2.10)
\]
Then, the disjoint sequence $(E_1^1, \ldots, E_1^{p_1}, E_2^1, \ldots, E_2^{p_2}, \ldots)$ does not satisfy (c).
(b)⇒(a). According to [6, Theorem I.2.4] there exists a countably additive measure \( \mu : \Sigma \to [0, \infty) \) so that

\[
\lim_{\mu(E) \to 0} |m_T| (E) = 0 \quad \text{uniformly in } T \in \mathcal{M}.
\]

(2.11)

Hence, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, if \( E \in \Sigma \) verifies \( \mu(E) < \delta \), then

\[
|m_T| (E) < \varepsilon / 2, \quad \text{for all } T \in \mathcal{H}.
\]

Next, given \((\varphi_n) \in \ell_1^w(\ell^w(\Omega))\) with \( \epsilon_1(\varphi_n) \leq 1 \), notice that the series \( \sum_{n=1}^{\infty} |\varphi_n(t)| \) is convergent for all \( t \in \Omega \). Put \( f_n(t) = \sum_{k=1}^{n} |\varphi_k(t)| \) and \( f(t) = \lim_{n \to \infty} f_n(t) \). By Egorov’s theorem, the sequence \((f_n)\) is quasi-uniformly convergent to \( f \). Then, there exists \( E \in \Sigma \) such that \( \mu(E) < \delta \) and

\[
f_{n|\Omega \setminus E} \to f|\Omega \setminus E
\]

(2.12)

uniformly. If \( C = \sup\{|m_T| (\Omega) : T \in \mathcal{M}\} \), there exists \( n_0 \in \mathbb{N} \) so that

\[
\sum_{n \geq n_0} |\varphi_n(t)| < \frac{\varepsilon}{2C}, \quad \forall t \in \Omega \setminus E.
\]

(2.13)

Now,

\[
\sum_{n \geq n_0} |T \varphi_n| = \sum_{n \geq n_0} \left\| \int_{\Omega} \varphi_n(t) \, dm_T \right\|
\]

\[
\leq \sum_{n \geq n_0} \left\| \int_{E} \varphi_n(t) \, dm_T \right\| + \sum_{n \geq n_0} \left\| \int_{\Omega \setminus E} \varphi_n(t) \, dm_T \right\|
\]

\[
\leq \sum_{n \geq n_0} \int_{E} |\varphi_n(t)| \, d|m_T| + \sum_{n \geq n_0} \int_{\Omega \setminus E} |\varphi_n(t)| \, d|m_T|
\]

\[
= \int_{E} \left( \sum_{n \geq n_0} |\varphi_n| \right) \, d|m_T| + \int_{\Omega \setminus E} \left( \sum_{n \geq n_0} |\varphi_n| \right) \, d|m_T|
\]

\[
\leq |m_T| (E) + \frac{\varepsilon}{2C} |m_T| (\Omega \setminus E)
\]

(2.14)

\[
< \varepsilon.
\]

We denote by \( \mathcal{V}(X,Y) \) the class of completely continuous operators from \( X \) into \( Y \), that is, the class of operators which map weakly convergent sequences in \( X \) into norm-convergent sequences in \( Y \). A set \( \mathcal{M} \subset \mathcal{V}(X,Y) \) is said to be uniformly completely continuous if, given a weakly convergent sequence \((x_n)\) in \( X \), \( (Tx_n) \) is norm convergent uniformly in \( T \in \mathcal{M} \). The following result gives some characterizations of uniformly completely continuous sets in \( \mathcal{V}(\ell^w(\Omega), Y) \). Recall that an operator \( T \) defined on \( \ell^w(\Omega) \) is completely continuous if and only if \( T \) is weakly compact [6, Corollary VI.2.17], so \( m_T \) is countably additive and regular, too.

**Theorem 2.5.** Let \( \mathcal{M} \subset \mathcal{V}(\ell^w(\Omega), Y) \) be a bounded set for the operator norm. The following statements are equivalent:

(a) \( \mathcal{M} \) is uniformly completely continuous,

(b) the family \( \{m_T : T \in \mathcal{M}\} \) is uniformly countably additive,
(c) \( M^* = \{ T^* : T \in M \} \) is collectively weakly compact, that is, the set \( \bigcup_{T \in M} T^*(B_{Y^*}) \) is relatively weakly compact in \( \mathcal{C}(\Omega)^* \).

**Proof.** (a)\( \Rightarrow \) (b). By [6, Proposition I.1.17], the family \( \{ m_T : T \in M \} \) is uniformly countably additive if and only if \( N = \{ y^* \circ m_T : T \in M, \ y^* \in B_{Y^*} \} \) is. According to [6, Lemma VI.1.13], we have to prove that

\[
\lim_{n \to \infty} y^* \circ m_T(O_n) = 0 \quad \text{uniformly in } N,
\]

for all disjoint sequences \( (O_n) \) of open subsets of \( \Omega \). By contradiction, suppose there exists such a sequence \( (O_n) \) for which \( \lim_{n \to \infty} y^* \circ m_T(O_n) = 0 \) but not uniformly in \( N \). Then, there exists \( \varepsilon > 0 \) and sequences \( (y^*_n) \subset B_{Y^*} \), \( (T_n) \in M \), and \( (O_{kn}) \subset (O_n) \) such that

\[
| y^*_n \circ m_{T_n}(O_{kn}) | > \varepsilon, \quad \forall \ n \in \mathbb{N}.
\]

Now, using the regularity of each \( m_{T_n} \), we can find a sequence of compact sets \( (H_n) \) with \( H_n \subset O_{kn} \) and

\[
\| m_{T_n} \|(O_{kn} \setminus H_n) < \frac{\varepsilon}{2}, \quad \forall \ n \in \mathbb{N},
\]

(\( \| m \| \) denotes the semivariation of \( m \), that is, \( \| m \|(E) = \sup \{ |y^* \circ m|(E) : y^* \in B_{Y^*} \} \)). By Urysohn’s lemma, for every \( n \in \mathbb{N} \) there exists a continuous function \( \varphi_n : \Omega \to [0,1] \) such that \( \varphi_n(H_n) = 1 \) and \( \varphi_n(\Omega \setminus O_{kn}) = 0 \). Obviously, the series \( \sum_{n=1}^{\infty} \varphi_n \) is unconditionally convergent in \( \mathcal{C}(\Omega) \). Since \( M \) is uniformly completely continuous, there exists \( n_0 \in \mathbb{N} \) such that

\[
\| T \varphi_n \| < \frac{\varepsilon}{2}, \quad \forall \ n \geq n_0, \ \forall \ T \in M.
\]

Then, we have

\[
\| m_{T_n}(O_{kn}) \| \leq \| m_{T_n}(O_{kn}) - T_n \varphi_n \| + \| T_n \varphi_n \|
\]

\[
= \left\| \int_{O_{kn}} \varphi_n \ dm_{T_n} - \int_{O_{kn}} \varphi_n \ dm_{T_n} \right\| + \| T_n \varphi_n \|
\]

\[
= \left\| \int_{O_{kn}} (1 - \varphi_n) \ dm_{T_n} \right\| + \| T_n \varphi_n \|
\]

\[
= \left\| \int_{O_{kn} \setminus H_n} (1 - \varphi_n) \ dm_{T_n} \right\| + \| T_n \varphi_n \|
\]

\[
\leq \| m_{T_n} \|(O_{kn} \setminus H_n) + \| T_n \varphi_n \|
\]

\[
< \varepsilon,
\]

for all \( n \geq n_0 \). This is in contradiction with (2.16).

(b)\( \Rightarrow \) (a). By [6, Theorem I.2.4], there exists a scalar countably additive measure \( \mu : \Sigma \to [0, \infty) \) such that \( \{ m_T : T \in M \} \) is uniformly \( \mu \)-continuous. Then, if \( (\varphi_n) \) is a sequence
that tends to zero weakly in \( \ell(\Omega) \), it is obvious that zero is the pointwise limit of the sequence \( (\varphi_n(t)) \). Now, using Egorov’s theorem and proceeding along similar lines as the proof of \((b)\Rightarrow(a)\) in Theorem 2.4, the proof concludes.

\[(b)\Leftrightarrow(c)\] The set \( \bigcup_{T \in M} T^*(B_{Y^*}) = \{ y^* \circ m_T : T \in M, \ y^* \in B_{Y^*} \} \subset \ell(\Omega)^* \) is relatively weakly compact if and only if it is bounded and uniformly countably additive [4, Theorem VII.13]. A call to [6, Proposition I.1.17] makes clear that \( \bigcup_{T \in M} T^*(B_{Y^*}) \) is uniformly countably additive if and only if only if condition \((b)\) is satisfied.\[\Box\]

**Corollary 2.6.** If \( M \subset \Pi_1(\ell(\Omega), Y) \) is uniformly 1-summing, then \( M \) is uniformly completely continuous.

The converse of the last result is not true in general.

**Proposition 2.7.** Suppose that the cardinal of \( \Omega \) is infinite. The following statements are equivalent:

(a) each subset of \( \Pi_1(\ell(\Omega), Y) \) uniformly completely continuous is uniformly 1-summing,

(b) \( Y \) is finite-dimensional.

**Proof.** \((a)\Rightarrow(b)\). By contradiction, suppose there is an unconditionally summable serie \( \sum_k y_k \) in \( Y \) such that \( \sum_k \| y_k \| = \infty \). Let \( (\omega_k) \) be a sequence in \( \Omega \) with \( \omega_k \neq \omega_l \) when \( k \neq l \). For each \( m \in \mathbb{N} \) consider the operator \( T_m : \ell(\Omega) \to Y \) defined by

\[ T_m\varphi = \sum_{k=1}^{m} \varphi(\omega_k)y_k. \tag{2.20} \]

It is not difficult to show that \( M = (T_m) \) is uniformly completely continuous. Nevertheless,

\[ \pi_1(T_m) = \sum_{k=1}^{m} \| y_k \| \xrightarrow{m, \infty} \infty, \tag{2.21} \]

so \( M \) cannot be uniformly 1-summing because it is not \( \pi_1 \)-bounded.

\((b)\Rightarrow(a)\). This follows easily in view of conditions \((b)\) in Theorems 2.4 and 2.5.\[\Box\]

We have showed that the converse of Corollary 2.6 is not true in general. However, a direct argument using Theorems 2.4 and 2.5 leads up to conclude that every uniformly completely continuous set \( M \subset \Pi_1(\ell(\Omega), Y) \) verifying the following condition is uniformly 1-summing:

(i) given \( T \in M \) and a finite subset \( \{(\varphi_1, y_1^*), \ldots, (\varphi_m, y_m^*)\} \) of \( \ell(\Omega) \times B_{Y^*} \), there exist \( S \in M \) and \( z^* \in B_{Y^*} \) such that \( \| y_n^* , T\varphi_n \| \leq \| z^* , S\varphi_n \| , \ n = 1, \ldots, m. \)

3. Final notes and examples. The Grothendieck-Pietsch domination theorem states that an operator \( T : X \to Y \) is \( p \)-summing if and only if there exists a positive Radon measure \( \mu \) defined on the (weak\(^*\)) compact space \( B_{X^*} \) such that

\[ \| Tx \|^p \leq \int_{B_{X^*}} | \langle x^*, x \rangle |^p \, d\mu(x^*), \tag{3.1} \]
for all $x \in X$ [5, Theorem 2.12]. Since the appearance of this theorem, there is a great interest in finding out the structure of uniformly $p$-dominated sets. A subset $\mathcal{M}$ of $\Pi_p(X,Y)$ is uniformly $p$-dominated if there exists a positive Radon measure $\mu$ such that the inequality (3.1) holds for all $x \in X$ and all $T \in \mathcal{M}$. In [3, 8, 9], the reader can find some of the most recent steps given on this subject. Now we are going to show that these sets are uniformly $p$-summing.

**Proposition 3.1.** If $\mathcal{M} \subset \Pi_p(X,Y)$ is a uniformly $p$-dominated set, then $\mathcal{M}^{**} = \{T^{**} : T \in \mathcal{M}\}$ is uniformly $p$-summing.

**Proof.** Let $\mu$ be a measure for which $\mathcal{M}$ is uniformly $p$-dominated. In a similar way as in the Pietsch factorization theorem [5, Theorem 2.13], we can obtain, for all $T \in \mathcal{M}$, operators $U_T : L_p(\mu) \to \ell_\infty(B_{Y^*})$, $\|U_T\| \leq \mu(B_{X^*})^{1/p}$, and an operator $V : X \to L_\infty(\mu)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow & & \downarrow i_Y \\
L_\infty(\mu) & \xrightarrow{i_p} & L_p(\mu) \\
\end{array}
\]

Here, $i_p$ is the canonical injection from $L_\infty(\mu)$ into $L_p(\mu)$ and $i_Y$ is the isometry from $Y$ into $\ell_\infty(B_{Y^*})$ defined by $i_Y(y) = ((y^*, y))_{y^* \in B_{Y^*}}$. Notice that $i_{p^{**}}$ can be viewed as $i_p$ composed with the canonical projection $P : L_\infty(\mu)^{**} \to L_\infty(\mu)$ which is simply the adjoint of the usual embedding $L_1(\mu) \to L_1(\mu)^{**}$. By weak compactness, we may and do consider $T^{**}$ as a map from $X^{**}$ into $Y$ for which

\[
i_Y \circ T^{**} = U_T \circ i_p \circ P \circ V^{**}.
\]

Given $\varepsilon > 0$ and $(x_n^{**}) \in \ell^p_\omega(X^{**})$, we can choose $n_0 \in \mathbb{N}$ so that

\[
\sum_{n \geq n_0} ||i_p \circ P \circ V^{**}(x_n^{**})||^p < \frac{\varepsilon}{\mu(B_{X^*})},
\]

because $i_p \circ P \circ V^{**}$ is $p$-summing. Then, we have

\[
\sum_{n \geq n_0} ||T^{**}x_n^{**}||^p = \sum_{n \geq n_0} ||i_Y \circ T^{**}(x_n^{**})||^p = \sum_{n \geq n_0} ||U_T \circ i_p \circ P \circ V^{**}(x_n^{**})||^p \\
\leq \mu(B_{X^*}) \sum_{n \geq n_0} ||i_p \circ P \circ V^{**}(x_n^{**})||^p < \varepsilon,
\]

for all $T \in \mathcal{M}$. So, $\mathcal{M}^{**}$ is uniformly $p$-summing. \qed
It is easy to show that the study of uniformly \( p \)-summing sets can be reduced to the behavior of its sequences. Indeed, a bounded set \( \mathcal{M} \) in \( \Pi_p(X,Y) \) is uniformly \( p \)-summing if and only if every sequence \( (T_n) \) in \( \mathcal{M} \) admits a uniformly \( p \)-summing subsequence. Thus, it seems to be interesting to make clear the relationship between uniformly \( p \)-summing sets and relatively weakly compact sets. For \( p = 1 \), we have the following result.

**Proposition 3.2.** Every relatively weakly compact set in \( \Pi_1(X,Y) \) is uniformly 1-summing.

**Proof.** Let \( \mathcal{M} \) be a relatively weakly compact set in \( \Pi_1(X,Y) \). Given \( \hat{x} = (x_n) \in \ell_w^1(X) \), consider the (weak-weak) continuous operator \( U_{\hat{x}} : \Pi_1(X,Y) \to \ell_a^1(Y) \) defined by \( U_{\hat{x}}(T) = (Tx_n) \). Then, \( U_{\hat{x}}(\mathcal{M}) \) is relatively weakly compact in \( \ell_a^1(Y) \); according to \([2, Theorem 2]\), we can conclude that \( \mathcal{M} \) is uniformly 1-summing. \( \square \)

Proposition 3.2 does not remain true if \( p = 2 \). For example, for each \( \beta = (\beta_n) \in \ell_2 \), consider the operator \( T_\beta : c_0 \to \ell_2 \) defined by \( T(\alpha_n) = (\alpha_n \cdot \beta_n) \) and put \( \mathcal{M} = \{ T_\beta : \beta \in B_\ell_2 \} \subset \Pi_2(c_0,\ell_2) \) \([5, Theorem 3.5]\). If we consider \( \ell_2 \) as a subspace of \( \Pi_2(c_0,\ell_2) \), the set \( \mathcal{M} = B_\ell_2 \) is relatively weakly compact. Nevertheless, no matter how we choose \( k \in \mathbb{N} \),

\[
\sum_{n \geq k} \| T_{\hat{x}_n} e_n \|^2 = 1,
\]

(3.6)

so \( \mathcal{M} \) cannot be uniformly 2-summing.

Now we show that there are uniformly \( p \)-summing sets failing to be relatively weakly compact.

**Proposition 3.3.** If every uniformly \( p \)-summing set is relatively weakly compact in \( \Pi_p(X,Y) \), then \( Y \) is reflexive.

**Proof.** Fixing \( x_0^* \in X^* \) with \( \| x_0^* \| = 1 \), the isometry \( y \mapsto x_0^* \otimes y \in x_0^* \otimes Y \) allows us to see \( Y \) as a subspace of \( \Pi_p(X,Y) \). If \( \varepsilon > 0 \) and \( (x_n) \in \ell_w^p(X) \), there exists \( n_0 \in \mathbb{N} \) so that

\[
\sum_{n \geq n_0} | \langle x_0^*, x_n \rangle |^p < \varepsilon;
\]

(3.7)

hence, for every \( y \in B_Y \),

\[
\sum_{n \geq n_0} \| (x_0^* \otimes y)(x_n) \|^p = \sum_{n \geq n_0} | \langle x_0^*, x_n \rangle |^p \| y \|^p < \varepsilon.
\]

(3.8)

This yields that \( B_Y \) is uniformly \( p \)-summing and, by hypothesis, weakly compact. \( \square \)

The converse of Proposition 3.3 is not always true. By contradiction, suppose every uniformly 1-summing set in \( \Pi_1(\ell_1,\ell_2) \) is relatively weakly compact. Because \( \ell_1 \) does not contain any copy of \( c_0 \), every bounded set in \( \Pi_1(\ell_1,\ell_2) \) is relatively weakly compact. Then, we conclude that \( \Pi_1(\ell_1,\ell_2) \) is reflexive, which is not possible since \( \ell_1^* \), viewed as a subspace of \( \Pi_1(\ell_1,\ell_2) \), is not.

However, if \( p = 1 \) and \( X = \ell(c_0) \), the reflexivity of \( Y \) is a sufficient condition for a uniformly 1-summing set to be relatively weakly compact. Indeed, if \( rbvca(\Sigma,Y) \) denotes
the set of all regular, countably additive, $Y$-valued measures $m$ on $\Sigma$ with bounded variation, recall that relatively weakly compact sets $M$ in $rbvca(\Sigma,Y)$ are those verifying the following conditions: (i) $M$ is bounded; (ii) the family of nonnegative measures $\{|m| : m \in M\}$ is uniformly countably additive; and (iii) for each $E \in \Sigma$, the set $\{m(E) : m \in M\}$ is relatively weakly compact in $Y$ [6, Theorem IV.2.5]. Having in mind the identification between $\Pi_1(\ell(\Omega),Y)$ and $rbvca(\Sigma,Y)$, and making use of the characterization of uniformly 1-summing sets obtained in Theorem 2.4, we conclude the next characterization.

**Corollary 3.4.** The following statements are equivalent:

(a) $Y$ is reflexive,

(b) every set $M$ in $\Pi_1(\ell(\Omega),Y)$ is uniformly 1-summing if and only if $M$ is relatively weakly compact.

It is well known that a linear operator $T$ is 1-summing if and only if $T^{**}$ is. So, it is natural to ask if a set $M$ is uniformly 1-summing whenever $M^{**} = \{T^{**} : T \in M\}$ is. Unfortunately, we are going to show that this is not true in general. It suffices to take $X$ as the separable $\ell_\infty$-space of Bourgain and Delbaen [1]. This space has the Radon-Nikodym property, so it does not contain any copy of $c_0$. Nevertheless, $X^*$ is isomorphic to $\ell_1$ and, therefore, $X^{**}$ contains a copy of $c_0$. Let $(e_n)$ be the canonical basis of $\ell_1$ and $J : \ell_1 \rightarrow X^*$ an isomorphism. Put $T_n = Je_n \in \Pi_1(X,\mathbb{R})$; the set $M = \{T_n : n \in \mathbb{N}\}$ is uniformly 1-summing since it is bounded and $X$ does not contain any copy of $c_0$. Notice that the elements of $M^{**}$ are the linear forms $x^{**} \in X^{**} \rightarrow \langle x^{**},Je_n \rangle \in \mathbb{R}$, for all $n \in \mathbb{N}$. If $(e_n^*)$ is the canonical basis of $c_0$, then $((J^*)^{-1}(e_n^*)) \in \ell_1^*(X^{**})$; hence, no matter how we choose $k \in \mathbb{N}$, it turns out that

$$
\sum_{n \geq k} \left| T_k^{**}((J^*)^{-1}(e_n^*)) \right| = \sum_{n \geq k} \left| \langle (J^*)^{-1}(e_n^*),Je_k \rangle \right| = \sum_{n \geq k} |\langle e_n^*,e_k \rangle| = 1, \quad (3.9)
$$

and $M^{**}$ cannot be uniformly 1-summing.

Nevertheless, if $M$ is a set of operators defined on $c_0$, then it is true that $M$ is uniformly 1-summing if and only if $M^{**}$ is too. To see this, notice that for a 1-summing operator $T : (\alpha_n) \in c_0 \rightarrow \sum_{n=1}^\infty \alpha_n x_n \in X$, the second adjoint $T^{**} : \ell_\infty \rightarrow X$ is defined by $T^{**}(\beta_n) = \sum_{n=1}^\infty \beta_n x_n$, for all $(\beta_n) \in \ell_\infty$.

When $M$ is a set of operators defined on a $\ell(\Omega)$-space, we do not know whether $M^{**}$ inherits the property or not. Anyway, we are going to prove the following weaker result. We inject isometrically $B(\Sigma)$ into $\ell(\Omega)^{**}$ in the natural way.

**Proposition 3.5.** If $M \subset \Pi_1(\ell(\Omega),X)$ is uniformly 1-summing, then $\tilde{M} = \{\tilde{T} : B(\Sigma) \rightarrow X : T \in M\}$ is uniformly 1-summing too.

**Proof.** The argument is similar to the one used in the proof of (b)⇒(a) in Theorem 2.4.

Finally, we give an example to show that Corollary 2.6 is not true if $\ell(\Omega)$ is replaced by a general Banach space $X$. It suffices to take $X = \ell_2$ and $M = \{e_n^* : n \in \mathbb{N}\}$, where $(e_n^*)$ is the unit basis of $\ell_2^* \simeq \ell_2$. The set $M$ is bounded in $\Pi_1(\ell_2,\mathbb{R})$ and, therefore, uniformly 1-summing but it is not uniformly completely continuous.
References


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