SOME RESULTS ON A GENERALIZED $\omega$-JACOBI TRANSFORM

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We introduce a generalized $\omega$-Jacobi transform and obtain images of certain functions under this transform. Moreover, we define a new probability density function (pdf) involving this new generalized $\omega$-Jacobi function. Some basic functions associated with the pdf, such as characteristic function, moments and distribution function, are evaluated.

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1. Introduction. Debnath [1] introduced the Jacobi transform of a function $g(x)$ defined in $-1 < x < 1$ by the integral

$$J[g(x)] = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}P_{n}^{(\alpha,\beta)}(x)g(x)dx,$$

where $P_{n}^{(\alpha,\beta)}$ is the Jacobi function of degree $n$ and orders $\alpha (> -1)$ and $\beta (> -1)$. Kalla et al. [3] have studied the integral

$$I_{a,b}^{\upsilon,\alpha,\beta} = \int_{-1}^{1} (1-x)^{a}(1+x)^{b}P_{\upsilon}^{(\alpha,\beta)}(x)dx,$$

with Re $a, \text{Re} b > -1$ and $P_{\upsilon}^{(\alpha,\beta)}$ is the Jacobi function, where

$$P_{\upsilon}^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_{\upsilon}}{\Gamma(\upsilon + 1)} \, _{2}F_{1}\left( -\upsilon, \upsilon + \lambda ; \frac{1-x}{2}, \alpha + 1 \right),$$

$\lambda = \alpha + \beta + 1$, and $_{2}F_{1}$ is the classical Gauss hypergeometric function and its partial derivatives with respect to $a$ and $b$. These results were extended by Sarabia [5] using the following integral,

$$I_{a,b,c,p}^{\upsilon,\alpha,\beta} = \int_{-1}^{1} (1-x)^{a}(1+x)^{b}P_{\upsilon}^{(\alpha,\beta,c,p)}(x)dx,$$

where $P_{\upsilon}^{(\alpha,\beta,c,p)}(x)$ is the generalized Jacobi function defined as

$$P_{\upsilon}^{(\alpha,\beta,c,p)}(x) = \frac{(\alpha + 1)_{\upsilon}}{\Gamma(\upsilon + 1)} \, _{3}F_{2}\left( -\upsilon, \upsilon + \lambda, c ; \frac{1-x}{2}, \alpha + 1, p \right),$$

with Re $(p - c - \beta) > 0$ and $_{3}F_{2}$ is generalized hypergeometric function [2].
Moreover, Sarabia and Kalla [6] defined and studied the generalized Jacobi transform as

$$J^{a,b,c,p}_{\alpha,\beta}[f(x),\nu] = \int_{-1}^{1} (1-x)^a(1+x)^b P_{\nu}(x) f(x) dx,$$

(1.6)

for continuous or sectionally continuous $f$ on $[-1,1]$. A number of integral transforms and their applications are given in [1, 7].

Throughout this sequel, we will use the following relation:

$$\int_{-1}^{1} (1-t)^n(1+t)^m dt = 2^{n+m+1} \int_{0}^{1} y^n(1-y)^m dy = 2^{n+m+1} B(n+1,m+1).$$

(1.7)

Further, we consider $\omega$-Kampe de Feriet function of two variables in the following form:

$$\omega \text{Kampe de Feriet function of two variables}.$$
where \( P_{\alpha,\beta,c,p}(x) \) is the generalized \( \omega \)-Jacobi function defined as

\[
P_{\alpha,\beta,c,p}(x) = \frac{(\alpha + 1)_{\nu}}{\Gamma(\nu + 1)} \left( 1 - x \right)^{\nu} P^\omega_{\nu,\frac{\nu}{2}} \left( -\nu, \nu + \lambda, c, \frac{1 - x}{2} \right),
\]

and \( \beta_2^\omega \) is the generalized \( \omega \)-Gauss hypergeometric function defined in [4, equation (1.9)]:

\[
\beta_2^\omega \left( \frac{a_1, a_2, a_3}{b_1, b_2}; y \right) = \frac{\Gamma(b_1)}{\Gamma(a_2)} \sum_{k=0}^{\infty} \frac{\Gamma(a_2 + \omega k)(a_1)_k(a_3)_k y^k}{\Gamma(b_1 + \omega k)(b_2)_k k!},
\]

where \( a \) is defined to be \( \Gamma(a + \omega k) / \Gamma(a) \).

For \( \omega = 1 \), (2.1) reduces to (1.6) which has been studied in [6]. In addition, if \( c = p \), (2.1) reduces to usual Jacobi transform [1].

Now we obtain images of some functions under the generalized \( \omega \)-Jacobi transform.

(1) For \( f(x) = 1 \), we have

\[
J_{\omega; \alpha, \beta}^{a,b,c,p} [1, \nu] = \int_{-1}^{1} (1 - x)^a (1 + x)^b P_{\alpha,\beta,c,p}(x) dx = \frac{(\alpha + 1)_{\nu}}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} B_k \int_{-1}^{1} (1 - x)^a (1 + x)^b dx,
\]

where

\[
B_k = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 + \omega k)} \frac{\Gamma(\nu + \lambda + \omega k)}{\Gamma(\nu + \lambda)} \frac{(-\nu)_k c_k}{(p)_k k!}.
\]

Using (1.7), we get

\[
J_{\omega; \alpha, \beta}^{a,b,c,p} [1, \nu] = \frac{2^{a+b+1} B(a + 1, b + 1)(\alpha + 1)_{\nu}}{\Gamma(\nu + 1)} \beta_3^\omega \left( -\nu, \nu + \lambda, c, a + 1 \right)_{\frac{\nu}{2}, a + b + 2; 1},
\]

where

\[
\beta_3^\omega \left( -\nu, \nu + \lambda, c, a + 1 \right)_{\frac{\nu}{2}, a + b + 2; 1} = \sum_{k=0}^{\infty} \frac{(a + 1)_k B_k}{(a + b + 2)_k}.
\]

(2) For \( f(x) = \ln(1 - x) \), we have

\[
J_{\omega; \alpha, \beta}^{a,b,c,p} [\ln(1 - x), \nu] = \int_{-1}^{1} \ln(1 - x)(1 - x)^a (1 + x)^b P_{\alpha,\beta,c,p}(x) dx = \frac{\partial}{\partial a} J_{\omega; \alpha, \beta}^{a,b,c,p} [1, \nu] = (\ln 2) J_{\omega; \alpha, \beta}^{a,b,c,p} [1, \nu] + \frac{2^{a+b+1} B(a + 1, b + 1)(\alpha + 1)_{\nu}}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} D_k,
\]

where

\[
D_k = \frac{(a + 1)_k B_k}{(a + b + 2)_k} \left[ \psi(a + k + 1) - \psi(a + b + k + 2) \right].
\]
In similar way we have, for \( f(x) = \ln(1 + x) \),

\[
J_{\omega;\alpha;\beta}^{a,b,c,p}[\ln(1 + x), \nu] = \int_{-1}^{1} \frac{\ln(1 + x)(1 - x)^{a}(1 + x)^{b} P_{\omega;\mu}^{(\alpha,\beta,c,p)}(x)}{dx}
\]

\[
= \frac{\partial}{\partial b} J_{\omega;\alpha;\beta}^{a,b,c,p}[1, \nu]
\]

\[
= \left[ \ln 2 + \psi(b + 1) \right] J_{\omega;\alpha;\beta}^{a,b,c,p}[1, \nu]
\]

\[
- \frac{2^{a+b+1}B(a+1,b+1)(\alpha+1)\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(a+1)kB_{k}}{(a+b+2)_k} \psi(a+k+1).
\]

(2.10)

where

\[
L_k = \frac{(a+1)kB_k}{(a+b+2)_k} \psi(a+b+k+2).
\]

(2.11)

(3) For \( f(x) = \ln((1 - x)/(1 + x)) \), using (2.8) and (2.10), we have

\[
J_{\omega;\alpha;\beta}^{a,b,c,p}\left[ \frac{1-x}{1+x} \right] = \psi(b + 1) \times J_{\omega;\alpha;\beta}^{a,b,c,p}[1, \nu]
\]

\[
- \frac{2^{a+b+1}B(a+1,b+1)(\alpha+1)\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(a+1)kB_{k}}{(a+b+2)_k} \psi(a+k+1).
\]

(2.12)

(4) For \( f(x) = (1 - x)^{A}(1 + x)^{B} \), we have

\[
J_{\omega;\alpha;\beta}^{a,b,c,p}[1 - x^A(1 + x)^B, \nu] = J_{\omega;\alpha;\beta}^{a+A,b+B,c,p}[1, \nu]
\]

\[
= \frac{2^{a+A+b+B+1}B(a+A+1,b+B+1)(\alpha+1)\nu}{\Gamma(\nu+1)}
\]

\[
\times R_{\nu}\left( -\nu, v+\lambda, c, a+1 \right)_{\nu}(\alpha+1, p, a+b+B+2, 1).
\]

(2.13)

(5) For \( f(x) = P_{\omega;\mu}^{y,\delta,d,q}(x) \), we have

\[
J_{\omega;\alpha;\beta}^{a,b,c,p}[P_{\omega;\mu}^{y,\delta,d,q}(x), \nu] = \frac{2^{a+b+1}B(a+1,b+1)(\alpha+1)\nu(y+1)\mu}{\Gamma(\nu+1)\Gamma(\mu+1)}
\]

\[
\times R_{1,2,3}\left( a+1, -\nu, v+\lambda, c, -\mu, \mu+\eta, d ; a+b+2, \alpha+1, p, \gamma+1, q ; 1, 1 \right)
\]

(2.14)

where \( \eta = y + \delta + 1 \) and

\[
\omega_{1,2,3;1,2,2}\left( a+1, -\nu, v+\lambda, c, -\mu, \mu+\eta, d ; a+b+2, \alpha+1, p, \gamma+1, q ; 1, 1 \right) = \sum_{n,k=0}^{\infty} \frac{(a+1)_{n+k}}{(a+b+2)_{n+k}} \times N_n \times B_k,
\]

(2.15)

with \( B_k \) given by (2.5) and

\[
N_n = \frac{\Gamma(y+1)\Gamma(\mu+\eta+\omega n)(-\mu)_{n}n!}{\Gamma(y+1+\omega n)\Gamma(\mu+\eta)(q)_n n!}, \quad \sigma_n^2 = E[X^2] - (E[X])^2.
\]

(2.16)
Remark 2.1. Letting $\omega = 1$ in previous results, we get same results presented in [6].

3. Generalized $\omega$-Jacobi random variable. Recently statistical distributions and their generalizations have played a significant role in application of applied statistics and reliability theory. In this section we define our $\omega$-Jacobi random variable and its pdf, then we derive some basic functions associated with this density function.

3.1. The probability density function. In a recent paper of Sarabia and Kalla [6], the following pdf has been studied:

$$h(x) = \frac{(1-x)^a(1+x)^b}{2^{a+b+1}B(a+1,b+1)} R^3 F_2 \left( \begin{array}{c} -v, u + \lambda, c \end{array} ; \begin{array}{c} 1-x \end{array} \right) \times 1[-1 \leq x \leq 1],$$  \hspace{1cm} (3.1)

where

$$R = 4 F_3 \left( \begin{array}{c} -v, u + \lambda, c + a + 1 \end{array} ; \begin{array}{c} \alpha + 1, p, a + b + 2 \end{array} \right).$$  \hspace{1cm} (3.2)

In the present paper, we introduce a generalization of (3.1) in the form given by (3.3). Using the condition $\int_{-1}^{1} g(t) \, dt = 1$, we obtained the pdf of a random variable $X$ associated with (1.2) to be given by

$$g(x) = g_{\omega,a,b,c,p;\omega}^{a,b,c,p;\omega}(x) = \frac{(1-x)^a(1+x)^b}{2^{a+b+1}B(a+1,b+1)} R^3 F_2 \left( \begin{array}{c} -v, u + \lambda, c \end{array} ; \begin{array}{c} 1-x \end{array} \right) \times 1[-1 \leq x \leq 1],$$  \hspace{1cm} (3.3)

where

$$R = 4 R^3 \left( \begin{array}{c} -v, u + \lambda, c + a + 1 \end{array} ; \begin{array}{c} \alpha + 1, p, a + b + 2 \end{array} \right).$$  \hspace{1cm} (3.4)

3.2. Some statistical functions. The aim of this section is to obtain some basic functions associated with the pdf $g(x)$, such as the population moments, the cumulative distribution function (cdf), and the survivor function.

3.2.1. The survivor and distribution functions. We derive the cdf $G(x)$ of the random variable $X$ after computing its survivor function $S(x)$.

Theorem 3.1. The survivor function $S(x)$ of the random variable $X$ is given by

$$S(x) = \frac{((1-x)/2)^{a+1}}{(a+1)B(a+1,b+1)} \omega^R 1^{1,3,1} R^1_{1,2,0} \left( \begin{array}{c} a+1; -v, u + \lambda, c; -b; \frac{1-x}{2} \end{array} ; \begin{array}{c} a+2; \alpha + 1, p; -; \frac{1-x}{2} \end{array} \right).$$  \hspace{1cm} (3.5)
Proof. Using (3.3) the survivor function \( S(x) \) of \( X \) becomes

\[
S(x) \triangleq P(X \geq x) = \int_x^1 g(t) dt
\]

\[
= 2 \int_0^{(1-x)/2} g(1-2y) dy = \frac{1}{\omega} \sum_{k=0}^{\infty} B_k \times \int_0^{(1-x)/2} y^{a+k}(1-y)^b dy
\]

\[
= \frac{1}{\omega} \sum_{k=0}^{\infty} B_{(1-x)/2} (a+k+1, b+1) \times B_k,
\]

where \( B_x(a, b) \) is the incomplete beta function [2].

\[
B_x(a, b) = \int_0^x y^a (1-y)^b dy = \frac{x^a}{a} \, _2F_1 \left( \frac{a, 1-b}{a+1}; x \right), \quad a, b > 0, \ 0 < x < 1,
\]

and \(_2F_1\) is Gauss hypergeometric function. Using this and (1.8) we get

\[
S(x) = \frac{((1-x)/2)^{a+1}}{(a+1)B(a+1, b+1) \omega} \sum_{n,k=0}^{\infty} \frac{(a+1)_{n+k} (-b)_n}{(a+2)_{n+k} n!} \left( \frac{1-x}{2} \right)^{n+k} \times B_k
\]

\[
= \frac{((1-x)/2)^{a+1}}{(a+1)B(a+1, b+1) \omega} R_{1,3,1}^3 \left( \frac{a+1}{a+2}; \frac{-\nu, \nu+\lambda, c; a+1}{\alpha+1, p, a+b+b+2}; 1 \right),
\]

which completes the proof.

Using the previous result, the cdf \( G(x) \) can be expressed as

\[
G(x) \triangleq P(X \leq x) = \int_{-1}^x g(t) dt = \int_{-1}^{1} g(t) dt - \int_{1}^x g(t) dt = 1 - S(x).
\]  

(3.9)

3.2.2. Population moments. In this subsection, we begin by evaluating the characteristic function, then we obtain the basic moments, such as the moment generating function, \( k \)th moment, and the mean. The following result will be used in obtaining the basic moments.

Theorem 3.2. For any \( A, B > 0 \),

\[
E[(1-X)^A(1+X)^B] = \frac{2^{A+B}B(a+1, b+1)}{B(a+1, b+1) \omega} \times \omega R_{3}^3 \left( \frac{-\nu, \nu+\lambda, c; a+1}{\alpha+1, p, a+b+b+2}; 1 \right),
\]

and, in particular,

\[
E[(1-X)^n] = \frac{2^n B(a+n+1, b+1)}{B(a+1, b+1) \omega} \times \omega R_{3}^3 \left( \frac{-\nu, \nu+\lambda, c; a+n+1}{\alpha+1, p, a+n+b+2}; 1 \right).
\]  

(3.11)
PROOF. Using (1.2) and (1.4), for \( K = (2^{a+b+1} B(a+1, b+1) R)^{-1} \), we have

\[
E[(1-X)^{A}(1+X)^{B}] = \int_{-1}^{1} (1-t)^{A}(1+t)^{B} g(t) dt
\]

\[
= K \int_{-1}^{1} (1-t)^{a+A}(1+t)^{b+B} \sum_{k=0}^{\infty} C_{k} \left( \frac{-v, u + \lambda, c; 1-t}{a+1, p; 1} \right) dt
\]

\[
= K \sum_{k=0}^{\infty} C_{k} \frac{B_{k}}{2^{k}} \int_{-1}^{1} (1-t)^{a+A+k}(1+t)^{b+B} dt
\]

\[
= K \sum_{k=0}^{\infty} 2^{a+A+k+b+B} B(a+A+k+1, b+B+1) \times B_{k}
\]

\[
= \frac{2^{A+B} B(a+1, b+B+1)}{B(a+1, b+B+1) R} \sum_{k=0}^{\infty} \left( \frac{a+A}{a+1, b+B+2} \right)^{k} B_{k}
\]

\[
= \frac{2^{A+B} B(a+1, b+B+1)}{B(a+1, b+B+1) R} \times 4 R_{3} \left( \frac{-v, u + \lambda, c, a+1}{a+1, p, a+B+2} \right)
\]

which completes the proof. \( \square \)

Now we state and prove our main theorem.

**Theorem 3.3.** The characteristic function of \( X \), for any real \( t \), is given by

\[
\varphi_{X}(t) = e^{it} \times R_{1,2,0}^{R_{1,3,0}} \left( \frac{a+1; -v, u + \lambda, c; -1}{a+b+2; \alpha+1, p; -1, -2it} \right).
\]

(3.13)

Its moment generating function of \( X \), for any real \( \tau \), is

\[
M_{X}(\tau) = e^{\tau} \times R_{1,2,0}^{R_{1,3,0}} \left( \frac{a+1; -v, u + \lambda, c; -1}{a+b+2; \alpha+1, p; -1, -2\tau} \right),
\]

(3.14)

and its \( r \)th moment is

\[
E[X^{r}] = \frac{1}{\omega R} \sum_{n=0}^{r} \binom{r}{n} (-2)^{n} (a+1)_{n} (a+b+2)_{n} \times 4 R_{3} \left( \frac{-v, u + \lambda, c, a+n+1}{a+1, p, a+n+b+2} \right),
\]

(3.15)

\[
E[X^{r}] = \frac{1}{\omega R} \Gamma(a+1) \sum_{n=0}^{\infty} \frac{(-v, u + \lambda, c, a+n+1)}{a+b+2} \times 4 R_{3} \left( \frac{-v, u + \lambda, c, a+n+1}{a+1, p, a+n+b+2} \right).
\]

(3.16)

**Special Case.** The mean, expected value of the random variable \( X \) is given by

\[
E[X] = 1 - \frac{2(a+1)}{a+1} \times R_{3} \left( \frac{-v, u + \lambda, c, a+2}{a+1, p, a+b+3} \right).
\]

(3.17)
**Proof.** We have

\[ \varphi_X(t) \triangleq E[e^{itX}] = E[e^{it(1-(1-X))}] = e^{it} \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} E[(1-X)^n]; \quad (3.18) \]

using (1.8) and (3.11),

\[ \varphi_X(t) = e^{it} \frac{\omega}{R} \sum_{n=0}^{\infty} \frac{(-2it)^n}{n!} \left( \frac{(a+1)n}{(a+b+2)n} \right)^\omega \times \mathcal{R}_3 \left( \frac{-v, v+\lambda, c, a+n+1}{\alpha+1, p, a+n+b+2}; 1 \right) \]

\[ = e^{it} \frac{\omega}{R} \times \sum_{n,k=0}^{\infty} \frac{(a+1)n+k}{(a+b+2)n+k} \frac{(-2it)^n}{n!} \times B_k \]

\[ = e^{it} \frac{\omega}{R} \times \mathcal{R}_{1,2,0} \left( \frac{-v, v+\lambda, c; -}{a+b+2; \alpha+1, p}; -; 1, -2it \right). \quad (3.19) \]

Observing that, with \( \tau = it \) in (3.13), we get moment generating function of \( X \). Indeed

\[ M_X(\tau) \triangleq E[e^{\tau X}] = e^{\tau} \frac{\omega}{R} \times \mathcal{R}_{1,2,0} \left( \frac{-v, v+\lambda, c; -}{a+b+2; \alpha+1, p}; -; 1, -2\tau \right). \quad (3.20) \]

To obtain the \( r \)th moment, we notice that for positive integer \( r \),

\[ E[X^r] = E[(1-(1-X))^r] = \sum_{n=0}^{r} \binom{r}{n} (-1)^n E[(1-X)^n]; \quad (3.21) \]

using (3.11),

\[ E[X^r] = \frac{1}{\omega} \sum_{n=0}^{r} \binom{r}{n} \frac{(-2)^n(a+1)n}{(a+b+2)n} \times \frac{\omega}{4} \mathcal{R}_3 \left( \frac{-v, v+\lambda, c, a+n+1}{\alpha+1, p, a+n+b+2}; 1 \right). \quad (3.22) \]

Now since the mean, expected value of the random variable \( X \) is a special case of this moment, namely, the mean is the 1st moment,

\[ E[X] = 1 - E[1-X] = 1 - \frac{2(a+1)}{(a+b+2)} \times \frac{\omega}{4} \mathcal{R}_3 \left( \frac{-v, v+\lambda, c, a+2}{\alpha+1, p, a+b+3}; 1 \right), \quad (3.23) \]

which completes the proof.

Similarly, we can obtain the variance of the random variable \( X \), \( \sigma_X^2 \), using (3.15) with \( k = 2 \), besides (3.17), since it is defined as

\[ \sigma_X^2 \triangleq E[X^2] - (E[X])^2. \quad (3.24) \]

**Remark 3.4.** Letting \( \omega = 1 \) in the previous theorem, we get the results presented in [6].
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