THE STRUCTURE OF A SUBCLASS OF AMENABLE BANACH ALGEBRAS

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We give sufficient conditions that allow contractible (resp., reflexive amenable) Banach algebras to be finite-dimensional and semisimple algebras. Moreover, we show that any contractible (resp., reflexive amenable) Banach algebra in which every maximal left ideal has a Banach space complement is indeed a direct sum of finitely many full matrix algebras. Finally, we characterize Hermitian \( \star \)-algebras that are contractible.

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1. Introduction. The purpose of this note is to establish the structure of some class of amenable Banach algebras. Let \( \mathcal{A} \) be a Banach algebra over the complex field \( \mathbb{C} \). We define a Banach left \( \mathcal{A} \)-module \( \mathcal{X} \) to be a Banach space which is also a unital left \( \mathcal{A} \)-module such that the linear map \( \mathcal{A} \times \mathcal{X} \to \mathcal{X} \), \((a,x) \to ax\), is continuous. Right modules are defined analogously. A Banach \( \mathcal{A} \)-bimodule is a Banach space with a structural \( \mathcal{A} \)-bimodule such that the linear map \( \mathcal{A} \times \mathcal{X} \times A \to \mathcal{X} \), \((a \times x \times b) \to axb\), is jointly continuous, where \( \mathcal{A} \times \mathcal{X} \times \mathcal{A} \) carries the Cartesian product topology. A submodule \( \mathcal{Y} \) of a Banach \( \langle \text{left, right, bi-} \rangle \mathcal{A} \)-module \( \mathcal{X} \) is a closed subspace of \( \mathcal{X} \) with the structural Banach \( \langle \text{left, right, or bi-} \rangle \mathcal{A} \)-module. A Banach left \( \mathcal{A} \)-module morphism \( \theta : \mathcal{X} \to \mathcal{Y} \) is a continuous linear map between two left Banach \( \mathcal{A} \)-modules such that \( \theta(ax) = a\theta(x) \) for all \( a \in \mathcal{A} \) and all \( x \in \mathcal{X} \). A Banach right \( \mathcal{A} \)-module morphism and a Banach \( \mathcal{A} \)-bimodule morphism are defined analogously. For each Banach \( \langle \text{left, bi-} \rangle \mathcal{A} \)-module \( \mathcal{X} \), the dual \( \mathcal{X}^* \) is naturally a Banach \( \langle \text{left, bi-} \rangle \mathcal{A} \)-bimodule with the module actions defined by \( \langle aT(x) = T(xa) \rangle, aT(x) = T(xa) \), and \( Ta(x) = T(ax) \), for all \( a \in \mathcal{A} \), \( T \in \mathcal{X}^* \), and \( x \in \mathcal{X} \), where \( T(x) \) denotes the evaluation of \( T \) at \( x \). If \( \mathcal{X} \), \( \mathcal{Y} \), and \( \mathcal{Z} \) are Banach \( \langle \text{left, or bi-} \rangle \mathcal{A} \)-modules and \( \theta : \mathcal{X} \to \mathcal{Y} \), \( \beta : \mathcal{Y} \to \mathcal{Z} \) are \( \langle \text{left, bi-} \rangle \) module morphisms, then the sequence

\[
\Sigma : 0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0
\]  

(1.1)

is exact if \( \theta \) is one-to-one, \( \mathcal{Y} \beta = \mathcal{Z} \), and \( \mathcal{Y} \theta = \ker \beta \). The exact sequence \( \Sigma \) is admissible if \( \beta \) has a continuous right inverse, equivalently, \( \ker \beta \) has a Banach space complement in \( \mathcal{Y} \). The admissible exact sequence splits if the right inverse of \( \beta \) is Banach \( \langle \text{left, bi-} \rangle \) module, equivalently, \( \ker \beta \) is a Banach space complement in \( \mathcal{Y} \) which is an \( \mathcal{A} \)-submodule.

A derivation from \( \mathcal{A} \) into a Banach \( \mathcal{A} \)-bimodule \( \mathcal{X} \) is a linear operator \( D : \mathcal{A} \to \mathcal{X} \) which satisfies \( D(ab) = D(a)b + aD(b) \), for all \( a, b \in \mathcal{A} \). Recall that for any \( x \in \mathcal{X} \), the mapping \( \delta_x : \mathcal{A} \to \mathcal{X} \) defined by \( \delta_x(a) = ax - xa \), \( a \in \mathcal{A} \), is a continuous derivation,
called an inner derivation. A Banach algebra \( \mathcal{A} \) is said to be contractible if for every Banach \( \mathcal{A} \)-bimodule \( \mathcal{X} \), each continuous derivation from \( \mathcal{A} \) into \( \mathcal{X} \) is inner. We say that \( \mathcal{A} \) is amenable whenever every continuous derivation from \( \mathcal{A} \) into \( \mathcal{X}^* \) is inner for each Banach \( \mathcal{A} \)-bimodule \( \mathcal{X} \). Obviously, every contractible Banach algebra is an amenable Banach algebra and the converse is true in the finite-dimension case. It is well known that a finite-dimensional algebra is semisimple if and only if it is isomorphic to a finite Cartesian product of a family of full matrix algebras. Using Theorem 2.1, it is easy to check that a finite Cartesian product of a family of full matrix algebras is contractible.

The purpose of this note is to contribute to the study of the following questions, raised, respectively, in [2], [3, page 817], and [5, page 212].

**Question 1.1.** Is every contractible Banach algebra semisimple?

**Question 1.2.** Is every reflexive amenable Banach algebra finite-dimensional and semisimple?

**Question 1.3.** Is every contractible Banach algebra finite-dimensional?

Recall that a Banach algebra is called a reflexive Banach algebra if it is reflexive as a Banach space. In this note, we will present two situations in which a contractible Banach algebra is finite-dimensional. First, we will give a partial answer to the above questions, where we assume that each maximal left ideal is complemented as a Banach space. This result improves [5, Proposition IV.4.3] for contractible Banach algebras and [3, Corollary 2.3] for reflexive amenable Banach algebras, where the authors suppose only that all of their primitive ideals have finite codimensions. Second, we will show that a Hermitian Banach \(*\)-algebra is contractible if and only if it is a finite-dimensional semisimple algebra.

2. Preliminaries. In this section, we recall some facts about the structure of contractible and amenable Banach algebras. Let \( \mathcal{A} \) be a Banach algebra over the complex field \( \mathbb{C} \) and let \( \mathcal{A}^{**} \) be the bidual of \( \mathcal{A} \) with the usual multiplication defined by \( \psi \cdot \phi(f) = \psi(f)\phi(f) \) for all \( \psi, \phi \in \mathcal{A}^{**} \) and \( f \in \mathcal{A}^* \). Consider on \( \mathcal{A}^{**} \) the Banach \( \mathcal{A} \)-bimodule structure defined by \( aT = \eta(a)T, Ta = T\eta(a) \) with \( \eta: \mathcal{A} \rightarrow \mathcal{A}^{**} \) the canonical map. Notice that if a Banach algebra \( \mathcal{A} \) has a bounded approximate identity, then its bidual \( \mathcal{A}^{**} \) has an identity. It is a fact that a contractible Banach algebra has an identity and an amenable Banach algebra admits bounded right, left, bilateral approximate identities. Of course, a reflexive amenable Banach algebra must be unital. We denote the identity element of \( \mathcal{A} \) by 1 and we write \( \mathcal{A} \hat{\otimes} \mathcal{A} \) for the completed projective tensorial product (see [4]). The Banach space \( \mathcal{A} \hat{\otimes} \mathcal{A} \) is a Banach \( \mathcal{A} \)-bimodule if we define

\[
a(b \otimes c) = ab \otimes c, \quad (b \otimes c)a = b \otimes ca, \quad a, b, c \in \mathcal{A}. \tag{2.1}
\]

For a unital Banach algebra \( \mathcal{A} \), a diagonal of \( \mathcal{A} \) is an element \( d \in \mathcal{A} \hat{\otimes} \mathcal{A} \) such that \( ad = da \), for all \( a \in \mathcal{A} \), and \( \pi(d) = 1 \), where \( \pi: \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A} \) is the canonical Banach \( \mathcal{A} \)-bimodule morphism. For such a Banach algebra \( \mathcal{A} \), a virtual diagonal of \( \mathcal{A} \) is an element
\( d \in (\mathcal{A} \otimes \mathcal{A})^{**} \) such that

\[
 ad = da, \quad \forall a \in \mathcal{A}, \quad \pi^{**}(d) = 1,
\]

where \( \pi^{**} : (\mathcal{A} \otimes \mathcal{A})^{**} \to \mathcal{A}^{**} \) is the bidual Banach \( \mathcal{A} \)-module morphism of \( \pi \). In the following theorems, we present characterizations of contractible (resp., amenable) Banach algebras. We recall, respectively, [1, Theorem 6.1] and [6, Theorem 1.3].

**Theorem 2.1.** Let \( \mathcal{A} \) be a Banach algebra. The following are equivalent:
(1) \( \mathcal{A} \) is contractible;
(2) \( \mathcal{A} \) has a diagonal.

**Theorem 2.2.** Let \( \mathcal{A} \) be a Banach algebra. The following are equivalent:
(1) \( \mathcal{A} \) is amenable;
(2) \( \mathcal{A} \) has a virtual diagonal.

We choose as a basis of the algebra \( M_n(\mathbb{C}) \) of all \( n \times n \) complex matrices the set of elementary matrices \( e_{ij} \). Consider \( d = \sum_{i,j} \delta_{ij} e_{ij} \otimes e_{ji} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \). Then \( Md = dM \), for all \( M \in M_n(\mathbb{C}) \), and \( \pi(d) = 1 \), where \( \pi : M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) is the canonical morphism. It follows that \( M_n(\mathbb{C}) \) is contractible.

Next, the following propositions hold.

**Proposition 2.3.** Let \( \mathcal{A} \) be a (contractible, amenable) Banach algebra. Then, if \( \theta : \mathcal{A} \to \mathcal{B} \) is a continuous homomorphism from \( \mathcal{A} \) into another Banach algebra \( \mathcal{B} \) with dense range, then \( \mathcal{B} \) is (contractible, amenable). In particular, if \( \mathcal{I} \) is a closed two-sided ideal of a (contractible, amenable) Banach algebra \( \mathcal{A} \), then \( \mathcal{A}/\mathcal{I} \) is (contractible, amenable) too.

**Proof.** Assume that \( \mathcal{A} \) is contractible. Let \( \mathcal{X} \) be a Banach \( \mathcal{B} \)-bimodule. Consider on \( \mathcal{X} \) the structure of \( \mathcal{A} \)-bimodule defined by \( a \cdot x = \theta(a)x \) and \( x \cdot a = x\theta(a) \). Since \( \theta \) is continuous, \( \mathcal{X} \) is a Banach \( \mathcal{A} \)-bimodule. Now, let \( D : \mathcal{B} \to \mathcal{X} \) be a continuous derivation. It is easy to see that \( D \circ \theta \) is a continuous derivation from \( \mathcal{A} \) to the Banach \( \mathcal{A} \)-bimodule \( \mathcal{X} \), and thus it is inner. Therefore, there exists \( x \in \mathcal{X} \) such that \( D(\theta(a)) = a \cdot x - x \cdot a = \theta(a)x - x\theta(a) \) for all \( a \in \mathcal{A} \). Since \( \theta(\mathcal{A}) \) is dense in \( \mathcal{B} \), we have \( D(b) = bx - xb \) for all \( b \in \mathcal{B} \). It follows that \( D \) is inner and \( \mathcal{B} \) is contractible. If \( \mathcal{A} \) is amenable, we will consider a continuous derivation \( D : \mathcal{B} \to \mathcal{X}^* \) from \( \mathcal{B} \) to the dual of the bimodule \( \mathcal{X} \) and we use the same way to prove that \( \mathcal{B} \) is amenable. \( \square \)

**Proposition 2.4** [1, Theorems 2.3 and 2.5]. Let \( \mathcal{A} \) be an amenable Banach algebra and let

\[
 \Sigma : 0 \to \mathcal{X}^* \to \mathcal{Y} \to \mathcal{X} \to 0
\]

be an admissible short exact sequence of Banach (left, right, or bi-) modules with \( \mathcal{X}^* \) a dual of \( \mathcal{X} \). Then \( \Sigma \) splits.

**Proposition 2.5** [1, Theorem 6.1]. Let \( \mathcal{A} \) be a contractible Banach algebra and let

\[
 \Sigma : 0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{X} \to 0
\]

be an admissible short exact sequence of Banach (left, right, or bi-) modules. Then \( \Sigma \) splits.
**Remark 2.6.** Notice that for each closed two-sided ideal $\mathcal{I}$ of a reflexive Banach algebra, $\mathcal{A}$ and the quotient $\mathcal{A}/\mathcal{I}$ are reflexive Banach algebras too.

**Proposition 2.7.** Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra and assume that $\mathcal{I}$ is a closed (left, two-sided) ideal of $\mathcal{A}$ which has a Banach space complement. Then there exists a closed (left, two-sided) ideal $\mathcal{J}$ of $\mathcal{A}$ such that

$$\mathcal{A} = \mathcal{I} + \mathcal{J}.$$  

**(2.5)**

**Proof.** Let $\mathcal{A}$ be an amenable Banach algebra and let $\mathcal{I}$ be a closed (left, two-sided) ideal of $\mathcal{I}$ which has a Banach space complement. Then the short exact sequence $\Sigma : 0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$ is admissible. If $\mathcal{A}$ is reflexive, then the space $\mathcal{I}$ will be the same, and so it will be the dual of the Banach (left, bi-) $\mathcal{A}$-module $\mathcal{I}^*$. By Proposition 2.4, $\Sigma$ splits and $\mathcal{I}$ has a Banach space complement which is a (left, two-sided) ideal. When $\mathcal{A}$ is contractible, by Proposition 2.5, we have the result. □

3. Main results

**Theorem 3.1.** Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra. Assume that each maximal left ideal of $\mathcal{A}$ is complemented as a Banach space in $\mathcal{A}$. Then there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$\mathcal{A} \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$  

**(3.1)**

**Proof.** By Section 2, the algebra $\mathcal{A}$ has an identity $1_{\mathcal{A}}$. Let $(\mathcal{M}_i)_{i \in I}$ be the family of all maximal left ideals. Since $\mathcal{M}_i$ is complemented as a Banach space for each $i$, there exists a left ideal $\mathcal{J}_i$ such that $\mathcal{A} = \mathcal{M}_i \oplus \mathcal{J}_i$. Notice that

$$\text{Rad}(\mathcal{A}) = \bigcap_i \mathcal{M}_i$$  

**(3.2)**

is the Jacobson radical of $\mathcal{A}$ and

$$\bigoplus_i \mathcal{J}_i \subseteq \text{Soc}(\mathcal{A}),$$  

**(3.3)**

where $\text{Soc}(\mathcal{A})$ is the socle of the algebra $\mathcal{A}$, that is, it is the sum of all minimal left ideals of $\mathcal{A}$ and it coincides with the sum of all minimal right ideals of $\mathcal{A}$. Recall that every minimal left ideal of $\mathcal{A}$ is of the form $\mathcal{A}e$, where $e$ is a minimal idempotent, that is, $e^2 = e \neq 0$ and $e \mathcal{A}e = \mathbb{C}e$. On the other hand, for each finite family of minimal idempotents $(e_k)_{k \in K}$, we have

$$\mathcal{A} = \bigoplus_{k \in K} \mathcal{A}e_k \bigoplus \bigcap_{k \in K} \mathcal{A}(1_{\mathcal{A}} - e_k).$$  

**(3.4)**

It follows from (3.3) and (3.4) that $\text{Soc}(\mathcal{A})$ is dense in $\mathcal{A}/\text{Rad}(\mathcal{A})$. This shows that $\mathcal{A}/\text{Rad}(\mathcal{A})$ is finite-dimensional. Therefore

$$\mathcal{A} = \text{Rad}(\mathcal{A}) \bigoplus \text{Soc}(\mathcal{A}).$$  

**(3.5)**
If $\text{Rad}(\mathcal{A}) \neq \{0\}$, this would mean that $\text{Rad}(\mathcal{A})$ has an identity, which is impossible. So, $\mathcal{A} = \text{Soc}(\mathcal{A})$, and then it is a finite direct sum of certain full matrix algebras.

**Corollary 3.2.** Every commutative ($\langle$ contractible, reflexive amenable $\rangle$) Banach algebra $\mathcal{A}$ is finite-dimensional and semisimple.

**Corollary 3.3.** Let $\mathcal{A}$ be a contractible or reflexive amenable Banach algebra such that every irreducible representation of $\mathcal{A}$ is finite-dimensional. Then $\mathcal{A}$ is finite-dimensional and semisimple.

**Proof.** It is easy to check that every primitive ideal of a Banach algebra is finite-codimensional if and only if each of its maximal left ideals is finite-codimensional. So, the corollary follows.

It should be emphasized that the following result appears in [9] or [5, Corollary in page 212].

**Corollary 3.4.** Every ($\langle$ contractible, reflexive amenable $\rangle$) $C^*$-algebra $\mathcal{A}$ is finite-dimensional and semisimple.

**Proof.** Suppose that $\mathcal{A}$ is a contractible or reflexive amenable $C^*$-algebra. Let $\mathcal{M}$ be a maximal left ideal. By [7, Theorems 5.3.5 and 5.2.4], the space $\mathcal{A}/\mathcal{M}$ is a Hilbert space. It follows that the short exact sequence

$$\Sigma: 0 \rightarrow \mathcal{M} \rightarrow \mathcal{A} \rightarrow \mathcal{A}/\mathcal{M} \rightarrow 0 \quad (3.6)$$

is admissible, and thus $\mathcal{M}$ has a Banach space complement. By Theorem 3.1, $\mathcal{A}$ is isomorphic to a finite direct sum of full matrix algebras.

**Remark 3.5.** Recall that a simple algebra is an algebra which has no proper ideals other than the zero ideal. To show that every ($\langle$ contractible, reflexive amenable $\rangle$) Banach algebra is finite-dimensional and semisimple, it suffices to prove that every ($\langle$ contractible, reflexive amenable $\rangle$) simple contractible Banach algebra is finite-dimensional. Indeed, let $\mathcal{A}$ be a contractible Banach algebra. Let $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$. Then the algebra $\mathcal{A}/\mathcal{P}$ is a ($\langle$ contractible, reflexive amenable $\rangle$) Banach algebra. Put $\mathcal{B} = \mathcal{A}/\mathcal{P}$ and consider some maximal two-sided ideal $\mathcal{M}$ of $\mathcal{B}$. Since $\mathcal{B}/\mathcal{M}$ is a ($\langle$ contractible, reflexive amenable $\rangle$) simple Banach algebra, it is finite-dimensional. There exists then a closed two-sided ideal $\mathcal{J}$ such that $\mathcal{B} = \mathcal{M} \oplus \mathcal{J}$. Recall that in a primitive algebra, every nonzero ideal is essential, that is, it has a nonzero intersection with every nonzero ideal of the algebra. It follow that $\mathcal{M} = 0$, and so $\mathcal{B}$ is finite-dimensional. Using Corollary 3.2, $\mathcal{A}$ must be a finite-dimensional and semisimple algebra. This completes the proof.

**Proposition 3.6.** Let $\mathcal{A}$ be a ($\langle$ contractible, reflexive amenable $\rangle$) simple contractible Banach algebra having a maximal left ideal complemented as a Banach space. Then $\mathcal{A}$ is finite-dimensional.

**Proof.** If $\mathcal{A}$ is an infinite-dimensional simple algebra, then $\text{Soc}(\mathcal{A}) = 0$. Moreover, if $\mathcal{A}$ is ($\langle$ contractible, reflexive amenable $\rangle$) with a maximal left ideal complemented as a Banach space, then $\mathcal{A}$ has a nontrivial minimal left ideal. This is a contradiction.
Now, assume that $\mathcal{A}$ is a unital Banach $*$-algebra which admits at least one state $\tau$. Then there exists a $*$-representation $\pi_{\tau}$ of $\mathcal{A}$ on a Hilbert space $H_{\tau}$, with a cyclic vector $\zeta$ of norm 1 in $H_{\tau}$ such that $\tau(a) = \langle \pi_{\tau}(a)\zeta, \zeta \rangle$, for all $a \in \mathcal{A}$, $\langle \cdot, \cdot \rangle$ being the inner product in $H_{\tau}$.

**Theorem 3.7.** A Hermitian Banach $*$-algebra $\mathcal{A}$ is contractible if and only if there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that (3.1) holds.

**Proof.** It suffices to show the “only if” part. Suppose that a Hermitian Banach algebra $\mathcal{A}$ is contractible. Let $T(\mathcal{A})$ be the set of all states of $\mathcal{A}$ and let $R^*(\mathcal{A})$ be the $*$-radical of $\mathcal{A}$, that is, the intersection of the kernels of all $*$-representations of $\mathcal{A}$ on Hilbert spaces. Since $\mathcal{A}$ is Hermitian and has an identity, $T(\mathcal{A}) \neq \emptyset$, and so $R^*(\mathcal{A}) \neq \emptyset$.

Put $\pi = \bigoplus_{\tau \in T(\mathcal{A})} \pi_{\tau}$ and $H = \bigoplus_{\tau \in T(\mathcal{A})} H_{\tau}$. Then $\pi$ is a $*$-representation of $\mathcal{A}$ on $H$. Consider

$$||\pi(a)|| = \sup_{\tau \in T(\mathcal{A})} ||\pi_{\tau}(a)||.$$ (3.7)

Then $\| \cdot \|$ is a C$^*$-norm on $\pi(A)$. Let $B$ denote the closure of $\pi(A)$, $\| \cdot \|$). Moreover, $\pi : \mathcal{A} \to B$ is a continuous mapping into a C$^*$-algebra $B$ such that $\ker(\pi) = R^*(\mathcal{A})$. As $\mathcal{A}$ is contractible, $B$ is also contractible. Using Corollary 3.4, the algebra $B$ has to be finite-dimensional. Notice that $\mathcal{A}/R^*(\mathcal{A})$ is isometric with the $*$-subalgebra $\pi(\mathcal{A})$ of $B$. Thus, it follows that $\mathcal{A}/R^*(\mathcal{A})$ is finite-dimensional. Since $R^*(\mathcal{A})$ is a finite-codimensional closed two-sided $*$-ideal, there exists a closed two-sided ideal $\mathcal{K}$ such that

$$\mathcal{A} = R^*(\mathcal{A}) \oplus \mathcal{K}.$$ (3.8)

Next, note that $\|\pi(a)\|^2 = \sup\{\tau(a^*a), \tau \in T(\mathcal{A})\} \geq |a^*a|_\sigma$, where $|a|_\sigma$ is the spectral radius of $a \in \mathcal{A}$. By Pták [8], we obtain $\|\pi(a)\|^2 \geq |a|_\sigma^2$. So, if $a \in R^*(\mathcal{A})$, then $|a|_\sigma = 0$. Therefore, every element of $R^*(\mathcal{A})$ is quasinilpotent. Notice that in general Rad($\mathcal{A}$) $\subseteq R^*(\mathcal{A})$. Since $R^*(\mathcal{A})$ is a closed two-sided $*$-ideal, we have $R^*(\mathcal{A}) = \text{Rad}(\mathcal{A})$, and so $\mathcal{A}$ is finite-dimensional and semisimple. \hfill $\square$

**REFERENCES**


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