ON A DIFFERENCE EQUATION WITH MIN-MAX RESPONSE

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We investigate the global behavior of the (positive) solutions of the difference equation

\[ x_{n+1} = \alpha_n + F(x_n, \ldots, x_{n-k}) , \]

where \((\alpha_n)\) is a sequence of positive reals and \(F\) is a min-max function in the sense introduced here. Our results extend several results obtained in the literature.

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1. Introduction. Let \(k\) be a positive integer and let \(\mathbb{R}_+\) be the set of all positive reals. We give the following definition.

**Definition 1.1.** A function \(F : \mathbb{R}_+^{k+1} \to \mathbb{R}_+\) is called a min-max function if it satisfies the inequality

\[
\frac{\land_{j=1}^{k+1} u_j}{\lor_{j=1}^{k+1} u_j} \leq F(u_1, u_2, \ldots, u_{k+1}) \leq \frac{\lor_{j=1}^{k+1} u_j}{\land_{j=1}^{k+1} u_j},
\]

for all \(u_j > 0, \ j = 1, \ldots, k+1\), where, as usual, the symbol \(\lor_{j=1}^{n} u_j\) stands for the maximum of the variables \(u_j, j = 1, \ldots, n\), and \(\land_{j=1}^{n} u_j\) stands for their minimum.

In Section 2, we give exact information on the form which a min-max function may have.

Simple examples of min-max functions are

\[
F_1(u_1, u_2) := \frac{u_2}{u_1}, \quad F_2(u_1, u_2) := \frac{u_1}{u_2}
\]

which appear as the response functions, respectively, in the difference equation

\[ y_{n+1} = \alpha + \frac{y_{n-1}}{y_n} \]

studied in [1] and in the difference equation

\[ y_{n+1} = \alpha + \frac{y_n}{y_{n-1}} \]

studied in [2]. These two equations have completely different behavior; see Remark 3.6. Also in [13, 14], the second author considered the closely related equation

\[ x_{n+1} = \alpha_n + \frac{x_{n-1}}{x_n} , \]
where \((\alpha_n)\) is either a periodic sequence (with period two) or a convergent sequence of nonnegative real numbers.

Motivated by the above-mentioned works, in this paper, we study the behavior of the difference equation

\[
x_{n+1} = \alpha_n + F(x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots,
\]

where the initial conditions \(x_{-k}, \ldots, x_0\) are positive real numbers, \((\alpha_n)\) is a sequence of positive real numbers, and \(F\) is a min-max function.

Since a min-max function takes the value 1 at the diagonal of the space \(\mathbb{R}^{k+1}_+\), it follows that in case the sequence \((\alpha_n)\) converges to a certain \(\alpha\), the positive real number

\[
K := \alpha + 1
\]

is the unique asymptotic equilibrium of (1.6).

Our purpose here is to discuss the boundedness and persistence of (1.6), as well as the attractivity of the asymptotic equilibrium \(\alpha + 1\), where \(\alpha\) is the limit of \((\alpha_n)\) whenever this exists. This follows immediately by Theorem 3.2, where we show that, if \(1 < \liminf \alpha_n \leq \limsup \alpha_n < +\infty\), then any solution \((x_n)\) satisfies the relation

\[
1 \leq \frac{\limsup x_n}{\liminf x_n} \leq \frac{\limsup \alpha_n - 1}{\liminf \alpha_n - 1}.
\]

Thus, if the sequence \((\alpha_n)\) converges to some \(\alpha(> 1)\), then any solution with positive initial values converges to the asymptotic equilibrium \(K = \alpha + 1\). This generalizes [1, Theorem 5.2] and part of [2, Theorem 1]. For the case \(\alpha_n = 1\), for all \(n\) (in Theorem 3.3), we show that any nonoscillatory solution converges to 2, while if \(F\) satisfies the additional (sufficient) conditions

\[
u_i < \vee_{j \neq i} u_j \Rightarrow F(u_1, u_2, \ldots, u_{k+1}) < \frac{\vee_{j \neq i} u_j}{u_i},
\]

\[
u_i > \wedge_{j \neq i} u_j \Rightarrow F(u_1, u_2, \ldots, u_{k+1}) > \frac{\wedge_{j \neq i} u_j}{u_i},
\]

then it is shown in Theorem 3.4 that all solutions converge to 2. Comparing this fact with the results in [1], we see that the pair of conditions (1.9)–(1.10) seems also to be necessary. Indeed, these conditions are not satisfied in case of (1.3) and, as it is shown in [1, Theorem 4.1], it has (nontrivial) solutions which are periodic with period 2.

In Theorem 3.5, we show that if \(\alpha_n = \alpha < 1\), for all \(n\), then there is a large class of equations of the form (1.6) which have unbounded (positive) solutions. This result extends [1, Theorem 3.1]. In the Section 4, we give two examples of difference equations with min-max response to illustrate our results.

Also the so-called \((2,2)\)-type equation defined in [6] (where about 50 types of difference equations are presented) includes the equation

\[
x_{n+1} = \frac{A_1 x_n + B_1 x_{n-1}}{A_2 x_n + B_2 x_{n-1}}.
\]
Under appropriate choice of the parameters, (1.11) can be written as

\[ x_{n+1} = \alpha + \frac{(\beta + \gamma)x_{n-1}}{\beta x_n + \gamma x_{n-1}}, \]  

which is of the type (1.6). Thus in this paper, we push further the investigation originated in [6] for such a form of (2,2)-type difference equations.

For other closely related results, which mostly deal with difference equations and inequalities whose response is (or it can be transformed into) a min-max function, see, for instance, [7, 8, 9, 10, 11, 12, 13, 14] and the references cited therein.

2. On the min-max functions. In this section, we give a characterization of min-max functions. The result is incorporated in the following theorem.

**Theorem 2.1.** A function \( F: \mathbb{R}^{k+1}_+ \to \mathbb{R}_+ \) is a min-max function if and only if there are nonnegative real-valued functions \( a_j(u_1, u_2, \ldots, u_{k+1}) \), \( b_j(u_1, u_2, \ldots, u_{k+1}) \), \( j = 1, 2, \ldots, k+1 \), such that

\[
\sum_{j=1}^{k+1} a_j(u_1, u_2, \ldots, u_{k+1}) = \sum_{j=1}^{k+1} b_j(u_1, u_2, \ldots, u_{k+1}) = 1,
\]

\[
F(u_1, u_2, \ldots, u_{k+1}) = \frac{\sum_{j=1}^{k+1} a_j(u_1, u_2, \ldots, u_{k+1}) u_j}{\sum_{j=1}^{k+1} b_j(u_1, u_2, \ldots, u_{k+1}) u_j},
\]

for all \( (u_1, u_2, \ldots, u_{k+1}) \in \mathbb{R}^{k+1}_+ \).

**Proof.** The “if” part is easily proved by using the form of \( F \) and the conditions on the coefficients \( a_j \), \( b_j \).

To show the inverse, assume that \( F(u_1, u_2, \ldots, u_{k+1}) \) is a min-max function and fix any element \( (u_1, u_2, \ldots, u_{k+1}) \in \mathbb{R}^{k+1}_+ \). We let

\[
v := \land_{j=1}^{k+1} u_j, \quad w := \lor_{j=1}^{k+1} u_j,
\]

thus \( v = u_{j_1} \) and \( w = u_{j_2} \), for two indices \( j_1, j_2 \in \{1, 2, \ldots, k+1\} \).

From the definition of the min-max functions, we know that the value \( F(u_1, u_2, \ldots, u_{k+1}) \) lies in the interval \([v/w, w/v]\), thus there is a number \( a \in [0,1] \) such that

\[
F(u_1, u_2, \ldots, u_{k+1}) = a \frac{w}{v} + (1-a) \frac{v}{w}.
\]

Let

\[
b := \frac{(1-a)v^2}{aw^2 + (1-a)v^2}.
\]

It is clear that \( b \) belongs to the interval \([0,1]\), and it depends on \( v \), \( w \) (thus on \( u_1, u_2, \ldots, u_{k+1} \)). By some simple calculations, we obtain

\[
(bw + (1-b)v)\left(\frac{aw}{v} + (1-a)\frac{v}{w}\right) = aw + (1-a)v
\]
and consequently we get
\[ F(u_1, u_2, \ldots, u_{k+1}) = a \frac{w}{v} + (1-a) \frac{v}{w} = \frac{aw + (1-a)v}{bw + (1-b)v}. \] (2.6)

This proves the theorem since we can set \( a_j(u_1, u_2, \ldots, u_{k+1}) = 0 \), if \( j \neq j_1, j_2 \), while \( a_{j_1}(u_1, u_2, \ldots, u_{k+1}) = 1-a \) and \( a_{j_2}(u_1, u_2, \ldots, u_{k+1}) = a \). Similar substitutions are used for the denominator. The proof is complete. \( \square \)

**Remark 2.2.** The quotient of any two elements of the class of all \( f : \mathbb{R}^{k+1}_+ \rightarrow \mathbb{R}_+ \) which satisfy an inequality of the form
\[ \wedge_{j=1}^{k+1} u_j \leq f(u_1, u_2, \ldots, u_{k+1}) \leq \vee_{j=1}^{k+1} u_j \] (2.7)
produces a min-max function.

3. The main results. Our first result refers to the boundedness of the solutions.

**Theorem 3.1.** Consider (1.6), where \( F \) is a min-max function and the sequence \((\alpha_n)\) satisfies
\[ 1 < C := \inf \alpha_n \leq \sup \alpha_n =: B < +\infty. \] (3.1)

Then any solution \((x_n)\) with positive initial values satisfies the condition
\[ \min \left\{ \wedge_{j=1}^{k+1} x_j, \frac{L C}{L-1} \right\} \leq x_n \leq L, \] (3.2)
for all \( n = 1, 2, \ldots \), where
\[ L := \max \left\{ \vee_{j=1}^{k+1} x_j, \frac{B C}{C-1} \right\}. \] (3.3)

Also, if \( \alpha_n = \alpha = 1 \), for all \( n \), then it holds that
\[ M \leq x_n \leq \frac{M}{M-1}, \] (3.4)
for all \( n \geq 1 \), where
\[ M := \min \left\{ \wedge_{j=1}^{k+1} x_j, \frac{\vee_{j=1}^{k+1} x_j}{\vee_{j=1}^{k+1} x_j - 1} \right\}. \] (3.5)

**Proof.** Let \( n > k+1 \). From (1.6), for all \( j \geq 1 \), we have
\[ C < x_j \leq \vee_{i=1}^{n} x_i. \] (3.6)
Also, for all $j = k + 2, k + 3, \ldots, n$, we get
\[
x_j \leq B + \frac{\vee_{i=j-k-1}^{j-1} x_i}{C} \leq B + \frac{\vee_{i=1}^{n} x_i}{C}.
\] (3.7)

These facts imply that
\[
\vee_{j=1}^{n} x_j \leq \max \left\{ \vee_{i=1}^{k+1} x_i, B + \frac{\vee_{i=1}^{n} x_i}{C} \right\},
\] (3.8)
from which we get
\[
x_n \leq \vee_{i=1}^{n} x_i \leq \max \left\{ \vee_{i=1}^{k+1} x_i, \frac{BC}{C - 1} \right\},
\] (3.9)
and therefore,
\[
C < x_m \leq L,
\] (3.10)
for all $m = 1, 2, \ldots$.

Next let $n > k + 1$. From (3.10) and (1.6), it follows that for all $j = k + 2, k + 3, \ldots, n$, it holds that
\[
x_j \geq C + \frac{\vee_{i=j-k-1}^{j-1} x_i}{L} \geq C + \frac{\vee_{i=1}^{n} x_i}{L}.
\] (3.11)

Therefore, we have
\[
x_j \geq \min \left\{ \vee_{i=1}^{k+1} x_i, C + \frac{\vee_{i=1}^{n} x_i}{L} \right\},
\] (3.12)
for all $j = 1, 2, \ldots$. This implies that
\[
\vee_{i=1}^{n} x_i \geq \min \left\{ \vee_{i=1}^{k+1} x_i, C + \frac{\vee_{i=1}^{n} x_i}{L} \right\}
\] (3.13)
and so
\[
\vee_{i=1}^{n} x_i \geq \min \left\{ \vee_{i=1}^{k+1} x_i, \frac{LC}{L - 1} \right\}.
\] (3.14)

This gives
\[
x_n \geq \vee_{i=1}^{n} x_i \geq \min \left\{ \vee_{i=1}^{k+1} x_i, \frac{LC}{L - 1} \right\},
\] (3.15)
which, together with (3.10), proves the first result of the theorem.
Next assume that \( \alpha_n = 1, n = 0, 1, \ldots \). To show inequality (3.4), we observe that
\[
M \leq x_n \leq \frac{M}{M-1},
\]
for all \( n = 1, 2, \ldots, k + 1 \). Also from (1.6), we get
\[
\begin{align*}
x_{k+2} &\geq 1 + \frac{\vee_{j=1}^{k+1} x_j}{\wedge_{j=1}^{k+1} x_j} \geq 1 + \frac{M}{M/(M-1)} = M, \\
x_{k+2} &\leq 1 + \frac{\vee_{j=1}^{k+1} x_j}{\wedge_{j=1}^{k+1} x_j} \leq 1 + \frac{M/(M-1)}{M} = \frac{M}{M-1}.
\end{align*}
\]
These arguments and the induction complete the proof.

**Theorem 3.2.** Consider (1.6), where \( F \) is a continuous min-max function and the sequence \((\alpha_n)\) satisfies the condition
\[
1 < \liminf \alpha_n \leq \limsup \alpha_n < +\infty.
\]
Then any (positive) solution \((x_n)\) satisfies relation (1.8). Hence, if the sequence \((\alpha_n)\) converges to some \( \alpha (> 1) \), then \((x_n)\) converges to (a constant, which, therefore, is equal to) \( \alpha + 1 =: K \).

**Proof.** Let \((x_n)\) be a solution. From Theorem 3.1, the solution is bounded, thus there are two-sided sequences, \((y_n)\) (upper full limiting sequence) and \((z_m)\) (lower full limiting sequence) of \((x_n)\) (see, e.g., [3, 4, 5]), satisfying (1.6), for all integers \( m \), and such that
\[
\liminf x_n = z_0 \leq z_m, \quad y_0 \leq \limsup x_n,
\]
for all \( m \). Let \( a_0 := \liminf \alpha_n \) and \( a^0 := \limsup \alpha_n \). Then from (1.6), we have
\[
y_0 \leq a^0 + \frac{y_0}{z_0}, \quad z_0 \geq a_0 + \frac{z_0}{y_0}.
\]
Combining these two relations, we obtain (1.8).

**Theorem 3.3.** Consider (1.6), where \( \alpha_n = 1, n = 0, 1, \ldots \), and \( F \) is a min-max function. Then every nonoscillatory (positive) solution converges to the equilibrium \( K = 2 \).

**Proof.** Assume first that \( x_n \geq 2 \), for all \( n \geq -k \). Set \( u_n := x_n - 2 \). From Theorem 2.1, we know that \( F \) may take the form (2.1), where the (nonnegative) functions \( a_j, b_j \) satisfy
\[
\sum_{j=1}^{k+1} a_j(x_n, \ldots, x_{n-k}) = \sum_{j=1}^{k+1} b_j(x_n, \ldots, x_{n-k}) = 1.
\]
Then we obtain
\[ u_{n+1} = \frac{\sum_{j=1}^{k+1} a_j(x_n,\ldots,x_{n-k})u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} - \frac{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} \]
\[ \leq \frac{\sum_{j=1}^{k+1} a_j(x_n,\ldots,x_{n-k})u_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} \leq \frac{1}{2} \vee_{n-k} u_j. \] (3.22)

Our intention is to show that \( \lim u_n = 0 \). To this end, we can either use [7, Lemma 1] or proceed as follows.

Let \((Y_m)\) be an upper full limiting sequence of \((u_n)\) with \( Y_m \leq Y_0 = \limsup u_n \), for all integers \( m \). Then, from the previous arguments, it follows that it satisfies the inequality
\[ Y_0 \leq \frac{1}{2} Y_0, \] (3.23)
thus we have \( Y_0 = 0 \). This and the fact that \( u_n \geq 0 \) imply that \( \lim x_n = 2 \).

Next, assume that \( x_n \leq 2 \), for all \( n \geq -k \). Set \( v_n := 2 - x_n \). From (1.3) and by using the form of the function \( F \), we obtain
\[ v_{n+1} = \frac{\sum_{j=1}^{k+1} a_j(x_n,\ldots,x_{n-k})v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} - \frac{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} \]
\[ \leq \frac{\sum_{j=1}^{k+1} a_j(x_n,\ldots,x_{n-k})v_{n+1-j}}{\sum_{j=1}^{k+1} b_j(x_n,\ldots,x_{n-k})x_{n+1-j}} \leq \frac{1}{M} \vee_{j=n-k} v_j, \] (3.24)

where \( M(>1) \) is the number defined in Theorem 3.1. By using this fact and following the same procedure as in the first case, we derive that \( \lim_{n \to \infty} v_n = 0 \), which implies that \( \lim x_n = 2 \), as desired. \( \square \)

**Theorem 3.4.** Consider (1.6), where \( \alpha_n = 1, \; n = 0, 1, \ldots, \) and \( F \) is a continuous min-max function satisfying the properties (1.9) and (1.10). Then every (positive) solution converges to the equilibrium \( K = 2 \).

**Proof.** Let \((x_n)\) be a solution. Then by Theorem 3.1, \((x_n)\) is bounded. Consider an upper full limiting sequence \((Y_m)\) and a lower full limiting sequence \((z_m)\) of \((x_n)\), as above. From (1.6), we have
\[ Y_0 \leq 1 + \frac{Y_0}{z_0}, \quad z_0 \geq 1 + \frac{z_0}{Y_0} \] (3.25)
and therefore, we get
\[ Y_0 z_0 = Y_0 + z_0. \] (3.26)
This gives
\[ \frac{1}{Y_0} + \frac{1}{z_0} = 1. \] (3.27)
If it happens that \( y_0, z_0 > 2\), or \( y_0, z_0 < 2\), then we should have \( \frac{1}{y_0}, \frac{1}{z_0} < \frac{1}{2} \) and \( \frac{1}{y_0}, \frac{1}{z_0} > \frac{1}{2} \), respectively. Both these arguments contradict (3.27). Therefore, we must have

\[
z_0 \leq 2 \leq y_0. \tag{3.28}
\]

Assume that there is some \( j \in \{-k-1, \ldots, -1\} \) such that \( y_j < y_0 \) and let \( j_0 \) be an index such that

\[
y_{j_0} = \bigwedge_{j=-k-1}^{1} y_j. \tag{3.29}
\]

Then from (1.9), we get

\[
y_{j_0} < \bigvee_{j \neq j_0} y_j \leq y_0 \tag{3.30}
\]

and so from (1.6) and condition (1.9), we have

\[
y_0 = 1 + F(y_{-1}, \ldots, y_{-k-1}) < 1 + \frac{y_0}{y_{j_0}} \leq 1 + \frac{y_0}{z_0}. \tag{3.31}
\]

This gives \( y_0 z_0 < y_0 + z_0 \), contradicting (3.26). Thus we have \( y_j = y_0 \), for all \( j = -k-1, \ldots, -1 \), and therefore,

\[
y_0 = 1 + F(y_{-1}, \ldots, y_{-k-1}) = 1 + F(y_0, \ldots, y_0) = 2. \tag{3.32}
\]

Similarly, we can use condition (1.10) to obtain \( z_0 = 2 \). The proof is complete. \( \square \)

Our final result refers to the case \( \alpha \in [0,1) \). We show that in this case, there are equations of the form (1.3) which admit unbounded solutions.

**Theorem 3.5.** Consider the equation

\[
x_{n+1} = \alpha + \frac{\sum_{i=0}^{m} a_i x_{n-2i-1}}{\sum_{i=0}^{m} b_i x_{n-2i}}, \tag{3.33}
\]

where \( m \in \mathbb{N}, \alpha \in [0,1) \), and where the coefficients \( a_j \) and \( b_j, j = 0, \ldots, m \), are nonnegative constants which satisfy the conditions

\[
\sum_{i=0}^{m} a_i = \sum_{i=0}^{m} b_i. \tag{3.34}
\]

Then there exist unbounded solutions of (3.33).

**Proof.** Obviously, without loss of the generality, we can assume that \( \sum_{i=0}^{m} a_i = \sum_{i=0}^{m} b_i = 1. \)
Assume that \( \alpha \in (0, 1) \). We choose the initial conditions such that
\[
 x_{-(2m+1)}, \ldots, x_{-1} > \frac{1}{1-\alpha} > 1 + \alpha, \quad \alpha < x_{-2m}, \ldots, x_0 < 1. \tag{3.35}
\]
We set
\[
 D := \wedge_{i=0}^{m} x_{-(2i+1)} \tag{3.36}
\]
and observe that
\[
 D > \frac{1}{1-\alpha}. \tag{3.37}
\]
From (3.33), we have
\[
 x_1 = \alpha + \frac{\sum_{i=0}^{m} a_i x_{-(2i+1)}}{\sum_{i=0}^{m} b_i x_{-2i}} > \alpha + \sum_{i=0}^{m} a_i x_{-(2i+1)} > \alpha + D, \\
 x_2 = \alpha + \frac{1}{\sum_{i=0}^{m} b_i x_{-2i}} \left( \frac{1}{b_0 x_1 + b_1 x_{-1} + \cdots + b_m x_{-2m+1}} \leq \alpha + \frac{1}{b_0 (\alpha + D) + (1- b_0)(1/(1-\alpha))} \right) \leq \alpha + \frac{1}{b_0 \alpha + 1/(1-\alpha)} \leq 1, \tag{3.38}
\]
\[
 x_3 = \alpha + \frac{\sum_{i=0}^{m} a_i x_{-2(2i+1)}}{\sum_{i=0}^{m} b_i x_{-2(2i)}} > \alpha + \sum_{i=0}^{m} a_i x_{-2(2i+1)} \\
 \geq \alpha + \min \{ x_1, x_{-1}, \ldots, x_{-2m+1} \} \geq \alpha + \min \{ x_1, x_{-1}, \ldots, x_{-2m-1} \} \\
 = \alpha + \min \{ x_1, \min \{ x_{-1}, \ldots, x_{-2m-1} \} \} \geq \alpha + D.
\]
Following the same procedure, we get
\[
 x_{2j+1} > \alpha + D, \quad x_{2j+2} < 1, \tag{3.39}
\]
for all \( j = 0, 1, \ldots, m \). By induction, we obtain
\[
 x_{(2m+2)j-(2s+1)} > \alpha^j + D, \tag{3.40}
\]
for all \( j \in \mathbb{N} \) and \( s = 0, 1, \ldots, m \), as well as
\[
 \alpha < x_{2n} < 1, \quad n = -m, -(m-1), \ldots, -1, \ldots. \tag{3.41}
\]
Inequality (3.40) implies the desired result in case \( \alpha > 0 \).
Assume that \( \alpha = 0 \). Choose \( \varepsilon \in (0, 1) \) and the initial conditions such that
\[
 x_{-(2m+1)}, \ldots, x_{-1} > \frac{1}{1-\varepsilon}, \quad 0 < x_{-2m}, \ldots, x_0 < 1 - \varepsilon. \tag{3.42}
\]
From (3.33), we have

\[ x_1 = \frac{\sum_{i=0}^{m} a_i x_{-2i} - b_0 x_{1-2i}}{\sum_{i=0}^{m} b_i x_{-2i} - b_0} > \frac{1/(1-\varepsilon)}{1-\varepsilon} = \frac{1}{(1-\varepsilon)^2} > \frac{1}{1-\varepsilon}, \]

\[ x_2 = \frac{\sum_{i=0}^{m} a_i x_{-2i} - b_0 x_{1-2i}}{\sum_{i=0}^{m} b_i x_{-2i} - b_0 - 2i} \leq \frac{1-\varepsilon}{b_0(1/(1-\varepsilon)^2) + (1-b_0)(1/(1-\varepsilon))} < 1-\varepsilon. \]

Following the same procedure, we get

\[ x_{2j+1} > \frac{1}{(1-\varepsilon)^2} > \frac{1}{1-\varepsilon}, \]

\[ x_{2j+2} < 1-\varepsilon, \]

for all \( j = 0, 1, \ldots, m \). By induction, we obtain

\[ x_{(2m+2)j-(2s+1)} > \frac{1}{(1-\varepsilon)^{j+1}}, \]

for all \( j \in \mathbb{N} \) and \( s = 0, 1, \ldots, m \), as well as

\[ 0 < x_{2n} < 1-\varepsilon, \quad n = 1, 2, \ldots. \]

From (3.45), the result follows.

**Remark 3.6.** Equation (3.33) includes the special case (1.3). Thus for \( \alpha \in (0, 1) \), Theorem 3.5 applies and therefore, (1.3) has unbounded solutions with positive initial values. On the other hand, (3.33) does not include the case (1.4) and as proved in [2], for the same values of \( \alpha \), (1.4) has a global attractor.

**Remark 3.7.** By some modifications of the proof of Theorem 3.5, we can prove the following result.

**Theorem 3.8.** Consider the equation

\[ x_{n+1} = \alpha_n + \frac{\sum_{i=0}^{m} a_i x_{n-2i-1}}{\sum_{i=0}^{m} b_i x_{n-2i}}, \]

where \( m \in \mathbb{N} \), \( (\alpha_n) \) is a sequence of positive real numbers such that \( \lim_{n \to \infty} \alpha_n =: A \in [0, 1) \), and where the coefficients \( a_j \) and \( b_j \), \( j = 0, \ldots, m \), are nonnegative constants which satisfy the conditions

\[ \sum_{i=0}^{m} a_i = \sum_{i=0}^{m} b_i. \]

Then there exist unbounded solutions of (3.47).
4. Some illustrative examples

**Example 4.1.** Consider the difference equation

\[ x_{n+1} = \alpha + \frac{\beta x_n + y x_n^2 + \delta x_n^2}{\beta x_n + y x_n x_{n-1} + \delta x_{n-1}^2}, \]

(4.1)

where all the coefficients are positive real numbers. The rational function on the right-hand side is a min-max function, since it can be written in the form

\[ \frac{((\beta + y x_n)/((\beta + y x_n + \delta x_{n-1}))x_n + ((\delta x_{n-1})/((\beta + y x_n + \delta x_{n-1}))x_{n-1}}{(\beta/((\beta + y x_n + \delta x_{n-1}))x_n + ((y x_n + \delta x_{n-1})/((\beta + y x_n + \delta x_{n-1}))x_{n-1}}. \]

(4.2)

Thus, from Theorems 3.2 and 3.4, we conclude that, for every fixed \( \alpha \geq 1 \), any solution of (4.1) converges to the equilibrium \( \alpha + 1 \). Notice that conditions (1.9) and (1.10) are also satisfied.

**Example 4.2.** Consider the difference equation

\[ x_{n+1} = \alpha + \frac{\sum_{j \in \{n,n-1,n-2\}} x_{j_1} x_{j_2} x_{j_3}}{x_{j_1}^2 + x_{j_2}^2 + x_{j_3}^2 + 6 x_n x_{n-1} x_{n-2}}, \]

(4.3)

where \( \alpha \geq 0 \). This is a third-order difference equation whose response on the right-hand side is a min-max function. Indeed, this can be written in the form

\[ \frac{\sum_{j \in \{n,n-1,n-2\}, j_1 \neq j_2 \neq j_3 \neq j_1} \left((x_{j_1}^2 + x_{j_2} x_{j_3} + x_{j_1} x_{j_2} x_{j_3})/ (x_n + x_{n-1} + x_{n-2})^2 \right) x_{j_1}}{\sum_{j \in \{n,n-1,n-2\}, j_1 \neq j_2 \neq j_3 \neq j_1} \left((x_{j_1}^2 + 2 x_{j_2} x_{j_3})/ (x_n + x_{n-1} + x_{n-2})^2 \right) x_{j_1}}. \]

(4.4)

Here, again, Theorems 3.2 and 3.4 apply and we conclude that in case \( \alpha \geq 1 \), any solution of (4.3) converges to \( \alpha + 1 \).

**References**


[5] G. L. Karakostas and S. Stević, *Slowly varying solutions of the difference equation* \( x_{n+1} = f(x_n,\ldots,x_{n-k}) + g(n,x_n,x_{n-1},\ldots,x_{n-k}) \), J. Difference Equ. Appl. 10 (2004), no. 3, 249–255.


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