ON DEPENDENT ELEMENTS IN RINGS

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Let \( R \) be an associative ring. An element \( a \in R \) is said to be dependent on a mapping \( F : R \to R \) in case \( F(x)a = ax \) holds for all \( x \in R \). In this paper, elements dependent on certain mappings on prime and semiprime rings are investigated. We prove, for example, that in case we have a semiprime ring \( R \), there are no nonzero elements which are dependent on the mapping \( \alpha + \beta \), where \( \alpha \) and \( \beta \) are automorphisms of \( R \).

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This research has been motivated by the work of Laradji and Thaheem [11]. Throughout, \( R \) will represent an associative ring with center \( Z(R) \). As usual the commutator \( xy - yx \) will be denoted by \( [x,y] \). We will use basic commutator identities \( [xy,z] = [x,z]y + x[y,z] \) and \( [x,yz] = [x,y]z + y[x,z] \). Recall that a ring \( R \) is prime if \( aRb = (0) \) implies that \( a = 0 \) or \( b = 0 \), and is semiprime if \( aRa = (0) \) implies \( a = 0 \). An additive mapping \( x \to x^* \) on a ring \( R \) is called involution in case \( (xy)^* = y^*x^* \) and \( x^{**} = x \) hold for all \( x,y \in R \). A ring equipped with an involution is called a ring with involution or \( \ast \)-ring. An additive mapping \( D : R \to R \) is called a derivation in case \( D(xy) = D(x)y +xD(y) \) holds for all pairs \( x,y \in R \). A derivation \( D \) is inner in case there exists \( a \in R \) such that \( D(x) = [a,x] \) holds for all \( x \in R \). An additive mapping \( T : R \to R \) is called a left centralizer in case \( T(xy) = T(x)y \) is fulfilled for all pairs \( x,y \in R \). This concept appears naturally in \( C^* \)-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would simply write that \( T : RR \to RR \) is a homomorphism of a right \( R \)-module \( R \) into itself. For any fixed element \( a \in R \), the mapping \( T(x) = ax \), \( x \in R \), is a left centralizer. In case \( R \) has the identity element \( T : R \to R \) is a left centralizer if and only if \( T \) is of the form \( T(x) = ax \), \( x \in R \), where \( a \in R \) is a fixed element. For a semiprime ring \( R \), a mapping \( T : R \to R \) is a left centralizer if and only if \( T(x) = qx \) holds for all \( x \in R \), where \( q \) is an element of Martindale right ring of quotients \( Q_x \) (see [1, Chapter 2]). An additive mapping \( T : R \to R \) is said to be a right centralizer in case \( T(xy) = xT(y) \) holds for all pairs \( x,y \in R \). In case \( R \) has the identity element \( T : R \to R \) is both left and right centralizer if and only if \( T(x) = qx \), \( x \in R \), where \( a \in Z(R) \) is a fixed element. In case \( R \) is a semiprime ring with extended centroid \( C \) a mapping \( T : R \to R \) is both left and right centralizer in case \( T(x) = \lambda x \), \( x \in R \), where \( \lambda \in C \) is a fixed element (see [1, Theorem 2.3.2]). For results concerning centralizers on prime and semiprime rings, we refer to [18, 19, 20, 21, 22, 23, 24]. Following [11] an element \( a \in R \) is said to be an element dependent on a mapping \( F : R \to R \) if \( F(x)a = ax \) holds for all \( x \in R \). A mapping \( F : R \to R \) is called a free action in case zero is the only element dependent
on $F$. It is easy to see that in semiprime rings there are no nonzero nilpotent dependent elements (see [11]). This fact will be used throughout the paper without specific references. Dependent elements were implicitly used by Kallman [10] to extend the notion of free action of automorphisms of abelian von Neumann algebras of Murray and von Neumann [14, 17]. They were later on introduced by Choda et al. [8]. Several other authors have studied dependent elements in operator algebras (see [6, 7]). A brief account of dependent elements in $W^*$-algebras has been also appeared in the book of Strătilă [16]. The purpose of this paper is to investigate dependent elements of some mappings related to derivations and automorphisms on prime and semiprime rings.

We will need the following two lemmas.

**Lemma 1** (see [2, Lemma 4]). *Let $R$ be a 2-torsion-free semiprime ring and let $a, b \in R$. If, for all $x \in R$, the relation $axb + bxa = 0$ holds, then $axb = bxa = 0$ is fulfilled for all $x \in R$.*

**Lemma 2** (see [12, Theorem 1]). *Let $R$ be a prime ring with extended centroid $C$ and let $a, b \in R$ be such that $axb = bxa$ holds for all $x \in R$. If $a \neq 0$, then there exists $\lambda \in C$ such that $b = \lambda a$.*

Our first result has been motivated by Posner’s first theorem [15] which states that the compositum of two nonzero derivations on a 2-torsion-free prime ring cannot be a derivation.

**Theorem 3.** *Let $R$ be a semiprime ring and let $D$ and $G$ be derivations of $R$ into itself. In this case the mapping $x \mapsto D^2(x) + G(x)$ is a free action.*

**Proof.** We have the relation

$$F(x)a = ax, \quad x \in R, \tag{1}$$

where $F(x)$ stands for $D^2(x) + G(x)$. A routine calculation shows that the relation

$$F(xy) = F(x)y + xF(y) + 2D(x)D(y) \tag{2}$$

holds for all pairs $x, y \in R$. Putting $xa$ for $x$ in (1) and using (2) we obtain $F(x)a^2 + xF(a)a + 2D(x)D(a)a = axa$, $x \in R$, which reduces because of (1) to

$$2D(x)D(a)a + xa^2 = 0, \quad x \in R. \tag{3}$$

Putting, in the above relation, $yx$ for $x$ and applying (3) we obtain $2D(y)xD(a)a = 0$, $x, y \in R$, whence it follows, putting $D(x)$ for $x$, that

$$2D(y)D(x)D(a)a = 0, \quad x, y \in R. \tag{4}$$

Multiplying relation (3) from the left by $D(y)$ and applying the above relation we obtain $D(y)x a^2 = 0$, $x, y \in R$, which gives, for $x = D(a)$ and $y = a$,

$$D(a)^2 a^2 = 0. \tag{5}$$
Multiplying relation (3) from the right by $a$, putting $x = a$ in (3), and applying the above relation we obtain $a^4 = 0$, which means that also $a = 0$. The proof of the theorem is complete.

For our next result, we need the concept of the so-called generalized derivations introduced by Brešar in [3]. An additive mapping $F : R \rightarrow R$, where $R$ is an arbitrary ring, is called a generalized derivation in case $F(xy) = F(x)y + xD(y)$ holds for all pairs $x, y \in R$, where $D : R \rightarrow R$ is a derivation. It is easy to see that $F$ is a generalized derivation if and only if $F$ is of the form $F = D + T$, where $D$ is a derivation and $T$ a left centralizer. For some results concerning generalized derivations, we refer the reader to [9].

**Theorem 4.** Let $F : R \rightarrow R$ be a generalized derivation, where $R$ is a semiprime ring, and let $a \in R$ be an element dependent on $F$. In this case $a \in Z(R)$.

**Proof.** We have the relation

$$F(x)a = ax, \quad x \in R. \tag{6}$$

Let $x$ be $xy$ in the above relation. Then we have

$$(F(x)y + xD(y))a = axy, \quad x, y \in R. \tag{7}$$

Using the fact that $F$ can be written in the form $F = D + T$, where $T$ is a left centralizer, we can replace $D(y)a$ by $F(y)a - T(y)a$ in (7), which gives, because of (6),

$$F(x)yF(x)a + [x, a]y - xT(y)a = 0, \quad x, y \in R. \tag{8}$$

Let $y$ be $yF(x)$ in (8). We have

$$F(x)yF(x)a + [x, a]yF(x) - xT(y)F(x)a = 0, \quad x, y \in R, \tag{9}$$

which reduces, according to (6), to

$$F(x)yax + [x, a]yx - xT(y)ax = 0, \quad x, y \in R. \tag{10}$$

Right multiplication of (8) by $x$ gives

$$F(x)yax + [x, a]yx - xT(y)ax = 0, \quad x, y \in R. \tag{11}$$

Subtracting (11) from (10) we arrive at

$$[x, a]y(F(x) - x) = 0, \quad x, y \in R. \tag{12}$$

Right multiplication of the above relation by $a$ gives, because of (6), $[x, a]y[x, a] = 0$, $x, y \in R$, whence it follows that $[x, a] = 0, x \in R$. The proof of the theorem is complete.

**Corollary 5.** Let $R$ be a semiprime ring and let $a, b \in R$ be fixed elements. Suppose that $c \in R$ is an element dependent on the mapping $x \mapsto ax + xb$. In this case $c \in Z(R)$. 
Proof. A special case of Theorem 4, since it is easy to see that the mapping \( x \mapsto ax + xb \) is a generalized derivation. □

In the theory of operator algebras the mappings \( x \mapsto ax + xb \), which we met in the above corollary, are considered as an important class of the so-called elementary operators (i.e., mappings of the form \( x \mapsto \sum_{i=1}^{n} a_i x b_i \)). We refer the reader to [13] for a good account of this theory.

Theorem 6. Let \( R \) be a noncommutative prime ring with extended centroid \( C \) and let \( a, b \in R \) be fixed elements. Suppose that \( c \in R \) is an element dependent on the mapping \( x \mapsto axb \). In this case the following statements hold:

1. \( bc \in Z(R) \);
2. \( abc = c \);
3. \( c = \lambda a \) for some \( \lambda \in C \).

Proof. We will assume that \( a \neq 0 \) and \( b \neq 0 \) since there is nothing to prove in case \( a = 0 \) or \( b = 0 \). We have

\[
(axb)c = cx, \quad x \in R.
\] (13)

Let \( x \) be \( xy \) in (13). Then

\[
(axyb)c = cxy, \quad x, y \in R.
\] (14)

According to (13) one can replace \( cx \) by \( (axb)c \) in the above relation. Then we have

\[
ax[bc, y] = 0, \quad x, y \in R,
\] (15)

which gives \( bc \in Z(R) \), which makes it possible to rewrite relation (13) in the form

\[
(abc - c)x = 0, \quad x \in R,
\] (16)

whence it follows that

\[
abc = c.
\] (17)

Putting \( xa \) for \( x \) in relation (13) we obtain, because of (17),

\[
axc = cxa, \quad x \in R,
\] (18)

whence it follows, according to Lemma 2, that \( c = \lambda a \) for some \( \lambda \in C \). The proof of the theorem is complete. □

Corollary 7. Let \( R \) be a noncommutative prime ring with the identity element and extended centroid \( C \) and let \( \alpha(x) = axa^{-1}, \) \( x \in R, \) be an inner automorphism of \( R \). An element \( b \in R \) is an element dependent on \( \alpha \) if and only if \( b = \lambda a \) for some \( \lambda \in C \).

Proof. According to Theorem 6 any element dependent on \( \alpha \) is of the form \( \lambda a \) for some \( \lambda \in C \). It is trivial to see that any element of the form \( \lambda a \), where \( \lambda \in C \), is an element dependent on \( \alpha \). □
We proceed to our next result.

**Theorem 8.** Let $R$ be a noncommutative 2-torsion-free prime ring and let $a, b \in R$ be fixed elements. Suppose that $c \in R$ is an element dependent on the mapping $x \rightarrow axb + bxa$. In this case the following statements hold:

1. $ac \in Z(R)$ and $bc \in Z(R)$;
2. $(ab + ba)c = c$;
3. $c^2 \in Z(R)$.

**Proof.** Similarly, as in the proof of Theorem 6, we will assume that $a \neq 0$ and $b \neq 0$. We have the relation

$$(axb + bxa)c = cx, \quad x \in R. \quad (19)$$

Let $x$ be $xy$ in the above relation. Then we have

$$(axyb + bxya)c = cxy, \quad x, y \in R. \quad (20)$$

Right multiplication of relation (19) by $y$ gives

$$(axb + bxa)c y = cxy, \quad x, y \in R. \quad (21)$$

Subtracting (21) from (20) we arrive at

$$ax[y, bc] + bx[y, ac] = 0, \quad x, y \in R. \quad (22)$$

Putting $cx$ for $x$ in the above relation we arrive at

$$acx[y, bc] + bcx[y, ac] = 0, \quad x, y \in R. \quad (23)$$

Now, multiplying the above relation first from the left by $y$, then putting $yx$ for $x$ in (23), and finally subtracting the relations so obtained from one another, we arrive at

$$y, ac]x[y, bc] + y, bc]x[y, ac] = 0, \quad x, y \in R. \quad (24)$$

Suppose that $ac \notin Z(R)$. In this case we have $[y, ac] \neq 0$ for some $y \in R$. Then it follows from relation (24) and Lemma 1 that $[y, bc] = 0$, which reduces relation (22) to $bx[y, ac] = 0, x, y \in R$, which means (recall that $b$ is different from zero) that $[y, ac] = 0$, contrary to the assumption. We have therefore $ac \in Z(R)$. Now relation (22) reduces to $ax[y, bc] = 0, x, y \in R$, whence it follows that $bc \in Z(R)$. Since $ac$ and $bc$ are in $Z(R)$, one can write relation (19) in the form $((ab + ba)c - c)x = 0, x \in R$, which gives

$$(ab + ba)c = c. \quad (25)$$

Putting $x = c$ in relation (19) we obtain

$$2(ac)(bc) = c^2. \quad (26)$$

Since $ac$ and $bc$ are both in $Z(R)$ it follows from the above relation that $c^2 \in Z(R)$. The proof of the theorem is complete. 

\[ \square \]
**Theorem 9.** Let \( R \) be a noncommutative 2-torsion-free prime ring with extended centroid \( C \) and let \( a, b \in R \) be fixed elements. In this case the mapping \( x \mapsto axb - bxa \) is a free action.

**Proof.** Again we assume that \( a \neq 0 \) and \( b \neq 0 \). Besides, we will also assume that \( a \) and \( b \) are \( C \)-independent, otherwise the mapping \( x \mapsto axb - bxa \) would be zero. We have the relation

\[(axb - bxa)c = cx, \quad x \in R. \tag{27}\]

Let \( x \) be \( xy \) in the above relation. Then we have

\[(axyb - bxya)c = cxy, \quad x, y \in R. \tag{28}\]

Right multiplication of relation (27) by \( y \) gives

\[(axb - bxa)c y = cxy, \quad x, y \in R. \tag{29}\]

Subtracting (29) from (28) we arrive at

\[ax[y, bc] - bx[y, ac] = 0, \quad x, y \in R. \tag{30}\]

Putting \( cx \) for \( x \) in the above relation we arrive at

\[acx[y, bc] - bcx[y, ac] = 0, \quad x, y \in R. \tag{31}\]

Now, multiplying first the above relation from the left by \( y \), then putting \( yx \) for \( x \) in (31), and finally subtracting the relations so obtained from one another, we arrive at

\[[y, ac]x[y, bc] - [y, bc]x[y, ac] = 0, \quad x, y \in R. \tag{32}\]

Suppose that \( ac \notin Z(R) \). In this case there exists \( y \in R \) such that \( [y, ac] \neq 0 \). Now it follows from the above relation and from Lemma 2 that

\[[y, bc] = \lambda_y[y, ac] \tag{33}\]

holds for some \( \lambda_y \in C \). According to (33) one can replace \( [y, bc] \) by \( \lambda_y[y, ac] \) in (30), which gives

\[(b - \lambda_y a)x[y, ac] = 0, \quad x \in R. \tag{34}\]

Since \( [y, ac] \neq 0 \) it follows from the above relation that \( b = \lambda_y a \), contrary to the assumption that \( a \) and \( b \) are \( C \)-independent. We have therefore proved that \( ac \in Z(R) \). Using this fact relation (30) reduces to

\[ax[y, bc] = 0, \quad x, y \in R, \tag{35}\]

whence it follows (recall that \( a \neq 0 \)) that \( bc \in Z(R) \). Since \( ac \) and \( bc \) are both in \( Z(R) \), one can rewrite relation (27) in the form \(((ab - ba)c - c)x = 0, x \in R\), which gives

\[(ab - ba)c = c. \tag{36}\]
Putting \( x = c \) in relation (27) and using the fact that \( bc \) is in \( Z(R) \), we obtain

\[
a[b,c]c = -c^2. \tag{37}
\]

From relation (36) one obtains, using the fact that \( ac \in Z(R) \),

\[
a[b,c] = c. \tag{38}
\]

Right multiplication of the above relation by \( c \) gives

\[
a[b,c]c = c^2. \tag{39}
\]

Comparing relations (37) and (39) one obtains \( c^2 = 0 \), since \( R \) is 2-torsion-free. Now it follows that \( c = 0 \), which completes the proof of the theorem. \( \square \)

**Theorem 10.** Let \( R \) be a semiprime ring and let \( \alpha \) and \( \beta \) be automorphisms of \( R \). In this case the mapping \( \alpha + \beta \) is a free action.

**Proof.** We have the relation

\[(\alpha(x) + \beta(x))a = ax, \quad x \in R. \tag{40}\]

Let \( x = xy \) in the above relation. Then

\[(\alpha(x)\alpha(y) + \beta(x)\beta(y))a = axy, \quad x,y \in R. \tag{41}\]

Replacing first \( ax \) by \( (\alpha(x) + \beta(x))a \) in the above relation and then \( ay \) by \( (\alpha(y) + \beta(y))a \), we arrive at

\[(\alpha(x)\alpha(y) + \beta(x)\beta(y))a = (\alpha(x) + \beta(x))(\alpha(y) + \beta(y))a, \quad x,y \in R, \tag{42}\]

which reduces to

\[\alpha(x)\beta(y)a + \beta(x)\alpha(y)a = 0, \quad x,y \in R. \tag{43}\]

The substitution \( zx \) for \( x \) in the above relation gives

\[\alpha(z)\alpha(x)\beta(y)a + \beta(z)\beta(x)\alpha(y)a = 0, \quad x,y,z \in R. \tag{44}\]

Left multiplication of (43) by \( \alpha(z) \) gives

\[\alpha(z)\alpha(x)\beta(y)a + \alpha(z)\beta(x)\alpha(y)a = 0, \quad x,y,z \in R. \tag{45}\]

Subtracting (44) from (45), we arrive at \( (\alpha(z) - \beta(z))\beta(x)\alpha(y)a = 0, \quad x,y,z \in R \). We therefore have

\[ (\alpha(z) - \beta(z))xya = 0, \quad x,y,z \in R. \tag{46}\]

Putting \( x = a \) and \( y(\alpha(z) - \beta(z)) \) for \( y \) in the above relation, we obtain \( (\alpha(z) - \beta(z))ax(\alpha(z) - \beta(z))a = 0, \quad x,z \in R \), whence it follows that

\[ \alpha(z)a = \beta(z)a, \quad z \in R. \tag{47}\]
According to (47) one can replace $\beta(y)a$ by $\alpha(y)a$ in (43), which gives $(\alpha(x) + \beta(x))\alpha(y)a = 0$, $x, y \in R$. We therefore have

$$(\alpha(x) + \beta(x))\alpha(y)a = 0, \quad x, y \in R. \tag{48}$$

Putting $y = a$ in the above relation and replacing $(\alpha(x) + \beta(x))a$ by $ax$, we obtain $axa = 0, x \in R$, which gives $a = 0$. The proof of the theorem is complete. \qed

The following question arises: what can be proved in case we have $\alpha - \beta$ instead of $\alpha + \beta$ in the above theorem? The mapping $\alpha - \beta$ is a special case of the so-called $(\alpha, \beta)$-derivations. An additive mapping $D : R \to R$, where $R$ is an arbitrary ring, is an $(\alpha, \beta)$-derivation if $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$ holds for all pairs $x, y \in R$, where $\alpha$ and $\beta$ are automorphisms of $R$. For results concerning $(\alpha, \beta)$-derivations, we refer the reader to [4, 5].

**Theorem 11.** Let $R$ be a semiprime ring and let $D : R \to R$ be an $(\alpha, \beta)$-derivation. In this case $D$ is a free action.

**Proof.** We have the relation

$$D(x)a = ax, \quad x \in R. \tag{49}$$

Putting $xy$ for $x$ in the above relation we obtain

$$D(x)\alpha(y)a + \beta(x)D(y)a = axy, \quad x, y \in R. \tag{50}$$

According to (49) one can replace $D(y)a$ by $ay$ in the above relation, which gives

$$D(x)\alpha(y)a + (\beta(x)a - ax)y = 0, \quad x, y \in R. \tag{51}$$

Putting $yz$ for $y$ in (51) we obtain

$$D(x)\alpha(y)\alpha(z)a + (\beta(x)a - ax)yz = 0, \quad x, y, z \in R. \tag{52}$$

On the other hand, right multiplication of (51) by $z$ gives

$$D(x)\alpha(y)az + (\beta(x)a - ax)yz = 0, \quad x, y, z \in R. \tag{53}$$

Subtracting (53) from (52) we obtain $D(x)\alpha(y)(\alpha(z)a - az) = 0, x, y, z \in R$. In other words, we have

$$D(x)y(\alpha(z)a - az) = 0, \quad x, y, z \in R. \tag{54}$$

The substitution $ay$ for $y$ in the above relation gives, because of (49),

$$axy(\alpha(z)a - az) = 0, \quad x, y, z \in R. \tag{55}$$

Putting $zx$ for $x$ in the above relation we obtain

$$azxy(\alpha(z)a - az) = 0, \quad x, y, z \in R. \tag{56}$$
Left multiplication of \((55)\) by \(\alpha(z)\) gives
\[
\alpha(z)axy(\alpha(z)a - az) = 0, \quad x, y, z \in R.
\] (57)

Subtracting (56) from (57) and multiplying the relation so obtained from the right-hand side by \(x\), we arrive at
\[
(\alpha(z)a - az)xy(\alpha(z)a - az)x = 0, \quad x, y, z \in R,
\] (58)

which gives first
\[
(\alpha(z)a - az)x = 0, \quad x, z \in R,
\] (59)

and then
\[
\alpha(z)a - az = 0, \quad z \in R.
\] (60)

Putting \(D(x)a\) instead of \(ax\) in (50), and \(ay\) for \(D(y)a\), we obtain
\[
D(x)(\alpha(y)a - ay) + \beta(x)ay = 0, \quad x, y \in R,
\]

which reduces because of (60) to \(\beta(x)ay = 0, \quad x, y \in R\), whence it follows that \(a = 0\). The proof of the theorem is complete. \(\square\)

**Corollary 12.** Let \(R\) be a semiprime ring and let \(\alpha\) and \(\beta\) be automorphisms of \(R\). In this case the mappings \(\alpha - \beta\) and \(ax - \beta a\), where \(a \in R\) is a fixed element, are free actions on \(R\).

**Proof.** According to Theorem 11 there is nothing to prove, since the mappings \(\alpha - \beta\) and \(ax - \beta a\) are \((\alpha, \beta)\)-derivations. \(\square\)

**Corollary 13.** Let \(R\) be a semiprime ring, let \(D : R \to R\) be a derivation, and let \(\alpha\) be an automorphism of \(R\). In this case the mappings \(x \mapsto D(\alpha(x)), x \mapsto \alpha(D(x)), x \mapsto D(\alpha(x)) + \alpha(D(x)), \) and \(x \mapsto D(\alpha(x)) - \alpha(D(x))\) are free actions.

**Proof.** A special case of Theorem 11, since all mappings are \((\alpha, \alpha)\)-derivations. \(\square\)

For our next result we need the following lemma.

**Lemma 14** (see [24, Lemma 1.3]). Let \(R\) be a semiprime ring and let \(a \in R\) be a fixed element. If \(a[x, y] = 0\) holds for all pairs \(x, y \in R\), then there exists an ideal \(I\) of \(R\) such that \(a \in I \subset Z(R)\).

**Proposition 15.** Let \(R\) be a semiprime ring and let \(\alpha : R \to R\) be an antiautomorphism. Suppose \(a \in R\) is an element dependent on \(\alpha\). In this case there exists an ideal \(I\) of \(R\) such that \(a \in I \subset Z(R)\). In case \(R\) is a prime ring, then either \(\alpha\) is a free action or \(\alpha\) is the identity mapping and \(R\) is commutative.

**Proof.** We have the relation
\[
\alpha(x)a = ax, \quad x \in R.
\] (61)
Putting $xy$ for $y$ in (61) and using (61) we obtain

$$axy = \alpha(xy)a = \alpha(y)\alpha(x)a = \alpha(y)ax = ayx.$$  \hfill (62)

We therefore have

$$a[x,y] = 0, \quad x,y \in R. \hfill (63)$$

From (63) and Lemma 14 it follows that there exists an ideal $I$ of $R$ such that $a \in I \subset Z(R)$, which completes the first part of the proof. The fact that $a \in Z(R)$ makes it possible to rewrite relation (61) in the form $(\alpha(x)-x)a = 0, x \in R$, whence it follows that

$$(\alpha(x)-x)y = 0, \quad x,y \in R.$$  \hfill (64)

In case $R$ is a prime ring it follows from the above relation that either $a = 0$ or $\alpha(x) = x$ for all $x \in R$, which completes the proof of the theorem. \hfill \Box

**Proposition 16.** Let $R$ be a semiprime $^*$-ring. Suppose that $a \in R$ is dependent on the involution. In this case there exists an ideal $I$ of $R$ such that $a \in I \subset Z(R)$ and $a^* = a$. In case $R$ is a prime ring, then either the involution is a free action or the involution is the identity mapping and $R$ is commutative.

**Proof.** Since all the assumptions of Proposition 15 are fulfilled, it remains to prove that $a^* = a$. Putting

$$x^*a = ax, \quad x \in R,$$  \hfill (65)

and $x = a$ in the relation we obtain $a^2 = a^*a$, which can be written in the form

$$(a-a^*)a = 0.$$  \hfill (66)

From the above relation we obtain, using the fact that $a \in Z(R)$,

$$0 = (a(a-a^*))^* = (a-a^*)a^*.$$  \hfill (67)

Thus we have

$$(a-a^*)a^* = 0.$$  \hfill (68)

Right multiplication of (66) by $x$ gives

$$(a-a^*)xa = 0, \quad x \in R,$$  \hfill (69)

since $a \in Z(R)$. Similarly, from (68) one obtains (note that also $a^* \in Z(R)$)

$$(a-a^*)xa^* = 0, \quad x \in R.$$  \hfill (70)
Subtracting (70) from (69), we obtain

\[(a - a^*)x(a - a^*) = 0, \quad x \in R,\]  

whence it follows that \(a^* = a\), which completes the proof. \(\square\)

**Theorem 17.** Let \(R\) be a semiprime ring and let \(\alpha\) be an antiautomorphism of \(R\). In this case the mapping \(x \mapsto \alpha(x) + x\) is a free action.

**Proof.** We have \((\alpha(x) + x)a = ax, \ x \in R\), which can be written in the form

\[\alpha(x)a = D(x), \quad x \in R,\]  

where \(D(x)\) stands for \([a,x]\). Putting \(xy\) for \(x\) in the above relation we obtain

\[\alpha(y)\alpha(x)a = D(x)y + xD(y), \quad x,y \in R.\]  

According to (72) one can replace \(\alpha(x)a\) by \(D(x)\) in the above relation, which gives

\[\alpha(y)D(x) = D(x)y + xD(y), \quad x,y \in R.\]  

Putting \(x = a\) in the above relation (note that \(D(a) = 0\)) one obtains

\[aD(y) = 0, \quad y \in R.\]  

According to (75), left multiplication of relation (72) by \(a\) reduces it to \(a\alpha(x)a = 0, \ x \in R\), whence it follows that \(a = 0\) by semiprimeness of \(R\), which completes the proof. \(\square\)

**Corollary 18.** Let \(R\) be a semiprime \(*\)-ring. The mapping \(x \mapsto x^* + x\) is a free action.

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**References**


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