We extend the Putnam-Fuglede theorem and the second-degree Putnam-Fuglede theorem to the nonnormal operators and to an elementary operator under perturbation by quasinilpotents. Some asymptotic results are also given.

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1. Introduction. Let $H$ be a complex Hilbert space and let $B(H)$ be the Banach algebra consisting of all the bounded linear operators on $H$. For the normal operators, we have the following well-known Putnam-Fuglede (PF) theorem [7].

**Theorem 1.1.** If $N, M$ are normal operators in $B(H)$, and if $X \in B(H)$ such that $NX = XM$, then $N^*X = XM^*$.

Putnam [7] also obtained another important result that we call the second-degree PF (SPF) theorem.

**Theorem 1.2.** If $N, M$ are normal operators in $B(H)$, and if $X \in B(H)$ such that $N(NX - XM) = (NX - XM)M$, then $NX = XM$.

If we let $A = (N_1, N_2)$ and $B = (M_1, M_2)$ denote tuples of commuting operators in $B(H)$, and define the elementary operators $\Delta_{(A,B)}$ and $\Delta_{(A^*,B^*)} \in B(B(H))$ by

\[
\Delta_{(A,B)}(X) = N_1 X N_2 - M_1 X M_2,
\]
\[
\Delta_{(A^*,B^*)}(X) = N_1^* X N_2^* - M_1^* X M_2^*,
\]

(1.1)

then an extension of the classical PF theorem, **Theorem 1.1**, is obtained as follows (see [4, 5]).

**Theorem 1.3.** If the operators $N_i, M_i \in B(H)$, $i = 1, 2$, are normal, then $\Delta_{(A,B)}(X) = 0$ for some $X \in B(H)$ implies $\Delta_{(A^*,B^*)}(X) = 0$.

Let $A = (N_1, N_2)$ and $B = (M_1, M_2)$. For $n = 2, 3, \ldots$, we define the high-order elementary operator $\Delta_{(A,B)}^{(n)}$ by

\[
\Delta_{(A,B)}^{(n)}(X) = \Delta_{(A,B)}(\Delta_{(A,B)}^{(n-1)}(X)), \quad X \in B(H).
\]

(1.2)

2. Putnam-Fuglede theorem under perturbation by quasinilpotents

**Theorem 2.1.** Let $A, B$ be normal operators, and let $C, D$ be quasinilpotents such that $AC = CA, BD = DB$. If $(A + C)X = X(B + D)$ for some $X \in B(H)$, then $AX = XB$. 

Proof. If \((A + C)X = X(B + D)\), then \(AX - XB = -(CX - XD)\). For any \(N, M \in B(H)\), denote by \(\delta_{NM}\) the linear operator on \(B(H)\):

\[
\delta_{NM}(X) = NX - XM;
\]

(2.1)

then \(\delta_{AB}(X) = -\delta_{CD}(X)\), so

\[
\delta^{(n)}_{AB}(X) = (-1)^n \delta^{(n)}_{CD}(X).
\]

(2.2)

Since \(\sigma(\delta_{CD}) = \sigma(C) - \sigma(D) = \{0\}\) (see [6]), we have \(n \sqrt{||\delta^{(n)}_{CD}(X)||} \to 0\). But

\[
\frac{n}{\sqrt{||\delta^{(n)}_{AB}(X)||}} \leq \frac{n}{\sqrt{||\delta^{(n)}_{CD}(X)||}} \frac{n}{\sqrt{||X||}},
\]

(2.3)

so \(n \sqrt{||\delta^{(n)}_{AB}(X)||} \to 0\). The theorem follows by a result of Anderson and Foiaş [1] which says that if \(A, B\) are normal operators, and \(n \sqrt{||\delta^{(n)}_{AB}(X)||} \to 0\), then \(AX - XB = 0\).

Remark 2.2. With the operators \(A\) and \(B\) being normal, it follows from Theorem 2.1 that \((A + C)X = X(B + D) \Rightarrow (A^* + C)X = X(B^* + D)\). It is, however, not true in general that \((A + C)^*X = X(B + D)^*\) (see [9]).

We give now a simple application of Theorem 2.1.

Corollary 2.3. Let \(N\) be a normal operator and let \(C\) be a quasinilpotent that commutes with \(N\). If \(f\) is a polynomial of degree \(n\) such that \(f(N + C) = 0\), then \(f^{(k)}(N)C^k = 0\) for \(k = 0, 1, \ldots, n\). So \(C\) is nilpotent of order at most \(n\). Moreover, if \(f\) has no multiple root, then \(C = 0\).

Proof. It is easy to see that

\[
f(N + C) = f(N) + f'(N)C + \frac{f''(N)}{2!} C^2 + \cdots + \frac{f^{(n)}(N)}{n!} C^n.
\]

(2.4)

Applying Theorem 2.1 to (2.4), we have \(f(N) = 0\) and

\[
f'(N)C + \frac{f''(N)}{2!} C^2 + \cdots + \frac{f^{(n)}(N)}{n!} C^n = 0,
\]

(2.5)

or

\[
\left(f'(N) + \frac{f''(N)}{2!} C + \cdots + \frac{f^{(n)}(N)}{n!} C^{n-1}\right)C = 0.
\]

(2.6)

Applying Theorem 2.1 again to (2.6) yields \(f'(N)C = 0\) and

\[
\left(\frac{f''(N)}{2!} + \cdots + \frac{f^{(n)}(N)}{n!} C^{n-2}\right)C^2 = 0.
\]

(2.7)

So we have \(f''(N)/2! C^2 = 0, \ldots, (f^{(n)}(N)/n!) C^n = 0\).

If \(f\) has no multiple root, then it follows from \(f(N) = 0\) that \(f'(N)\) is invertible. As \(f'(N)C = 0\), we know immediately that \(C = 0\).
Lemma 2.4. Let $C, M \in B(H)$. If $C$ is quasinilpotent, then the only solution $X \in B(H)$ of $X = CXM$ is $X = 0$.

Proof. If $X = CXM$, we have, for $n = 2, 3, \ldots$, $X = C^n XM^n$, so

$$\|X\| \leq \|C^n\| \|X\| \|M^n\| \leq \|C^n\| \|X\| \|M\|^n.$$  \hfill (2.8)

But with $C$ being quasinilpotent, it follows that

$$\frac{n}{\sqrt{n}} \|C^n\| \|M^n\|^n \to 0, \quad n \to \infty.$$  \hfill (2.9)

Thus $\|C^n\| \|M\|^n \to 0$, so $X = 0$ by (2.8).

Lemma 2.5. Let $N$ be a normal operator and let $C, D$ be quasinilpotents such that $N, C, D$ mutually commute. If $M \in B(H)$, and $(N + C)X(N + C) = MXD$ for some $X \in B(H)$, then $NXN = 0$.

Proof. Suppose that $X \in B(H)$ such that $(N + C)X(N + C) = MXD$. If the kernel $\text{Ker}(N) \neq \{0\}$, then letting $P$ be the project from $H$ to $\text{Ker}(N)$, we have $NPXN = 0$, $NXPN = 0$. Therefore, to prove $NXN = 0$, it is sufficient to prove $NP\perp X\perp N = 0$. Thus we can assume that $\text{Ker}(N) = \{0\}$. Let

$$N = \int_{\sigma(N)} \lambda dE_\lambda$$  \hfill (2.10)

be the spectral decomposition of $N$. Define $\Delta_\epsilon = \{z \mid |z| \leq \epsilon\}$, $\Delta_\epsilon^c = C \setminus \Delta_\epsilon$, and $T_\epsilon = E(\Delta_\epsilon^c)T|_{E(\Delta_\epsilon)H}$ for any $T \in B(H)$, then we have

$$(N_\epsilon + C_\epsilon)X_\epsilon(N_\epsilon + C_\epsilon) = M_\epsilon X_\epsilon D_\epsilon,$$  \hfill (2.11)

but $N_\epsilon$ is invertible, so

$$(N_\epsilon + C_\epsilon)^{-1} = N_\epsilon^{-1} + C_\epsilon^o,$$  \hfill (2.12)

where $C_\epsilon^o$ is also quasinilpotent, and

$$X_\epsilon = (N_\epsilon + C_\epsilon)^{-1} M_\epsilon X_\epsilon D_\epsilon (N_\epsilon + C_\epsilon)^{-1}.$$  \hfill (2.13)

Because $D_\epsilon(N_\epsilon + C_\epsilon)^{-1}$ is quasinilpotent, by Lemma 2.4, we have $X_\epsilon = 0$. Letting $\epsilon \to 0$, we have $X = 0$, so $NXN = 0$. This completes the proof.

Lemma 2.6. Let $N$ be a normal operator and let $C$ be quasinilpotent such that $NC = CN$. If $(N + C)X(N + C) = X$ for some $X \in B(H)$, then $NXN = X$.

Proof. If $\text{Ker}(N) \neq \{0\}$, then let $P$ be the project $H \to \text{Ker}(N)$. If $(N + C)X(N + C) = X$ for some $X \in B(H)$, then $P(N + C)X(N + C) = PX$, so $CPX(N + C) = PX$, but since $C$ is quasinilpotent, by Lemma 2.4, we have $PX = 0$. The same way shows that $XP = 0$. Therefore, we may assume $\text{Ker}(N) = \{0\}$.
Let \( N = \int_{\sigma(N)} \lambda dE_\lambda \) be the spectral decomposition of \( N \). Define \( \Delta_\epsilon, \Delta'_\epsilon, \) and \( T_\epsilon \) to be the same as in Lemma 2.5. Then

\[
(N_\epsilon + C_\epsilon)X_\epsilon(N_\epsilon + C_\epsilon) = X_\epsilon
\]  

(2.14)

or

\[
(N_\epsilon + C_\epsilon)X_\epsilon = X_\epsilon (N_\epsilon + C_\epsilon)^{-1} = X_\epsilon (N_{\epsilon^{-1}} + C_\epsilon'),
\]

(2.15)

where \( C_\epsilon' \) is quasinilpotent. So by Theorem 2.1, \( N_\epsilon X_\epsilon = X_\epsilon N_{\epsilon^{-1}} \), or \( X_\epsilon = N_\epsilon X_\epsilon N_\epsilon \). Letting \( \epsilon \to 0 \), we have \( NN = X \). \( \square \)

Using the same technique as in the proof of Lemma 2.6, we are able to obtain the following theorem.

**THEOREM 2.7.** Let \( N, M \) be normal operators and let \( C, D \) be quasinilpotents such that \( NC = CN \) and \( MD = DM \). If \( (N + C)X(N + C) = (M + D)X(M + D) \) for some \( X \in B(H) \), then \( NN = MM \).

**Proof.** If \( \text{Ker}(N) \neq \{0\} \), then let \( P \) be the project: \( H \to \text{Ker}(N) \). If \( (N + C)X(N + C) = (M + D)X(M + D) \) for some \( X \in B(H) \), then \( P(N + C)X(N + C) = P(M + D)X(M + D) \), that is, \( CPX(N + C) = (M + D)PX(M + D) \). Since \( C \) is quasinilpotent, by Lemma 2.5, we have \( MPXM = 0 \). The same method shows that \( MXPM = 0 \). Therefore, we can assume that \( \text{Ker}(N) = \{0\} \).

Let \( N = \int_{\sigma(N)} \lambda dE_\lambda \) be the spectral decomposition of \( N \). Define \( \Delta_\epsilon, \Delta'_\epsilon, \) and \( T_\epsilon \) to be the same as in Lemma 2.5. Then

\[
(N_\epsilon + C_\epsilon)X_\epsilon(N_\epsilon + C_\epsilon) = (M_\epsilon + D_\epsilon)X_\epsilon(M_\epsilon + D_\epsilon).
\]

(2.16)

If we write \( (N_\epsilon + C_\epsilon)^{-1} = N_{\epsilon^{-1}} + C_\epsilon' \), where \( C_\epsilon' \) is quasinilpotent, then the above equation becomes

\[
X_\epsilon = (N_{\epsilon^{-1}} + C_\epsilon')(M_\epsilon + D_\epsilon)X_\epsilon(M_\epsilon + D_\epsilon)(N_{\epsilon^{-1}} + C_\epsilon')
\]

(2.17)

or

\[
X_\epsilon = (N_{\epsilon^{-1}}M_\epsilon + F_\epsilon)X_\epsilon(N_{\epsilon^{-1}}M_\epsilon + F_\epsilon),
\]

(2.18)

where \( F_\epsilon \) is quasinilpotent. Applying Lemma 2.6 to the equation yields \( X_\epsilon = N_{\epsilon^{-1}}M_\epsilon X_\epsilon N_{\epsilon^{-1}}M_\epsilon \) or \( N_\epsilon X_\epsilon N_\epsilon = M_\epsilon X_\epsilon M_\epsilon \). Letting \( \epsilon \to 0 \), we have \( NN = MM \). \( \square \)

More generally, using Berberian’s trick, we obtain the PF theorem under perturbation by quasinilpotents for the elementary operators.

**THEOREM 2.8.** Let \( N_1, N_2, M_1, M_2 \) be normal operators and let \( C_1, C_2, D_1, D_2 \) be quasinilpotents such that \( N_i, M_i, C_i, D_i \) mutually commute for \( i = 1, 2 \). If \( (N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2) \) for some \( X \in B(H) \), then \( N_1 M_2 = M_1 X M_2 \).
Proof. Let
\[ \tilde{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \]
(2.19)
where \( T = N, M, C, D \); then \( \tilde{N}, \tilde{M} \) are normal, and \( \tilde{C}, \tilde{D} \) are quasinilpotents in \( B(H \oplus H) \). If
\( (N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2) \), then \( (\tilde{N} + \tilde{C})\tilde{X}(\tilde{N} + \tilde{C}) = (\tilde{M} + \tilde{D})\tilde{X}(\tilde{M} + \tilde{D}) \), so \( \tilde{N}\tilde{X}\tilde{N} = \tilde{M}\tilde{X}\tilde{M} \) by Theorem 2.7, that is, \( N_1XN_2 = M_1XM_2 \).

3. Second-degree PF theorem. First we will extend Theorem 1.2 to the more general case.

**Theorem 3.1.** Let \( N_1, N_2, M_1, M_2 \) be normal operators such that \( N_1M_1 = M_1N_1, N_2M_2 = M_2N_2 \). If \( N_1(N_1XN_2 - M_1XM_2)N_2 = M_1(N_1XN_2 - M_1XM_2)M_2 \) for some \( X \in B(H) \), then \( N_1XN_2 - M_1XM_2 = 0 \).

**Proof.** First we will prove that if \( N, M \) are normal operators, then \( N(NXN - MXM)N = M(NXN - MXM)M \) implies \( NXN = MXM \).

If \( \text{Ker}(N) \neq \{0\} \), then letting \( P \) be the project \( H \rightarrow \text{Ker}(N) \), we have \( PN(NXN - MXM)N = PM(NXN - MXM)M \). That is, \( 0 = -M^2PX^2 \) or \( M(M(PX^2) - (PX^2)0) = (M(PX^2) - (PX^2)0)0 \). By the SPF theorem (Theorem 1.2), \( MPXM = 0 \). By the same way, we have \( MPXM = 0 \). Similarly, \( MXPM = 0 \). So we may assume that \( \text{Ker}(N) = \{0\} \).

Let \( T_\epsilon \) be the same as in Lemma 2.5. I f \( X \in B(H) \) such that
\[ N(NXN - MXM)N = M(NXN - MXM)M, \]
then
\[ N_\epsilon(N_\epsilon X_\epsilon N_\epsilon - M_\epsilon X_\epsilon M_\epsilon)N_\epsilon = M_\epsilon(N_\epsilon X_\epsilon N_\epsilon - M_\epsilon X_\epsilon M_\epsilon)M_\epsilon \]
(3.2)
or
\[ X_\epsilon - N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon = N_\epsilon^{-1}M_\epsilon(X_\epsilon - N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon)N_\epsilon^{-1}M_\epsilon. \]
(3.3)
Since \( N_\epsilon^{-1}M_\epsilon \) is normal, by [2], we have
\[ X_\epsilon - N_\epsilon^{-1}M_\epsilon X_\epsilon N_\epsilon^{-1}M_\epsilon = 0 \]
(3.4)
or
\[ N_\epsilon X_\epsilon N_\epsilon = M_\epsilon X_\epsilon M_\epsilon. \]
(3.5)
Letting \( \epsilon \rightarrow 0 \), we have \( NXN = MXM \).

In general, let
\[ \tilde{N} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}. \]
(3.6)
If
\[ N_1(N_1XN_2 - M_1XM_2)N_2 = M_1(N_1XN_2 - M_1XM_2)M_2, \]  
then
\[ \tilde{N}(\tilde{N}\tilde{X}\tilde{N} - \tilde{M}\tilde{X}\tilde{M})\tilde{N} = \tilde{M}(\tilde{N}\tilde{X}\tilde{N} - \tilde{M}\tilde{X}\tilde{M})\tilde{M}; \]  
so \( \tilde{N}\tilde{X}\tilde{N} = \tilde{M}\tilde{X}\tilde{M} \), that is, \( N_1XN_2 = M_1XM_2 \).

Let \( A = (N_1, N_2) \), \( B = (M_1, M_2) \) be tuples of commuting operators in \( B(H) \). We say that \((A, B)\) has the SPF theorem if for any \( X \in B(H) \) and for some \( n \geq 2 \) such that \( \Delta_{(A,B)}^{(n)}(X) = 0 \), we have \( \Delta_{(A,B)}(X) = 0 \).

**Theorem 3.2.** Let \( N, M, D \in B(H) \) such that \( N \) commutes with \( D \) and \( M \). If \( N \) is invertible and \( D \) is quasinilpotent, then \( ((N, N), (M, D)) \) has the SPF theorem.

**Proof.** If
\[ N(NXN - MXD)N = M(NXN - MXD)D, \]  
then
\[ X - N^{-1}MXN^{-1}D = N^{-1}M(X - N^{-1}MXN^{-1}D)N^{-1}D. \]  
Note that \( N^{-1}D \) is quasinilpotent; so by applying Lemma 2.4 to \( X - N^{-1}MXN^{-1}D \), we have \( X - N^{-1}MXN^{-1}D = 0 \), that is, \( NXN - MXD = 0 \).

**Theorem 3.3.** Let \( N, M \in B(H) \) such that \( N \) commutes with \( M \). If \( M \) is invertible and \( \|N\|\|M^{-1}\| \leq 1 \), then \( ((N, N), (M, M)) \) has the SPF theorem.

**Proof.** If \( (3.1) \) holds for some \( X \in B(H) \), then
\[ NM^{-1}XNM^{-1} - X = NM^{-1}(NM^{-1}XNM^{-1} - X)NM^{-1}. \]  
Since \( \|N\|\|M^{-1}\| \leq 1 \), by [2], we have \( NM^{-1}XNM^{-1} - X = 0 \), that is, \( NXN = MXM \).

The next theorem establishes the relationship between the SPF theorem and the PF theorem under perturbation by nilpotents.

**Theorem 3.4.** Let \( N_i, M_i \in B(H) \) and let \( C_i, D_i \) be nilpotents such that \( C_i, D_i, N_i, M_i \) mutually commute for \( i = 1, 2 \). If \( ((N_1, N_2), (M_1, M_2)) \) has the SPF theorem, then \( (N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2) \) implies that \( N_1XN_2 = M_1XM_2 \).

**Proof.** If
\[ (N_1 + C_1)X(N_2 + C_2) = (M_1 + D_1)X(M_2 + D_2), \]  
then
then by expanding both sides of the equation and moving $M_1 XM_2$ to the left-hand side and moving all the terms in the left-hand side to the right-hand side except $N_1 XN_2$, we have

$$N_1 XN_2 - M_1 XM_2 = S(X),$$

where $S$ is a linear operator on $B(H)$ defined by

$$S(X) = -N_1 X C_2 - C_1 X N_2 - C_1 X C_2 + M_1 X D_2 + D_1 X M_2 + D_1 X D_2.$$  

It is clear that $S^{(2)}(X) = S(S(X))$ consists of $6^2$ terms like

$$(-1)^{s_1 s_2} M_1^{r_1} M_2^{r_2} C_1^{s_1} C_2^{s_2} D_1^{t_1} X N_2^{m_2} m_2 C_2^{s_2} D_2^{t_2},$$

where $s_1 + t_1 + s_2 + t_2 \geq 2, \ldots$,  

$S^{(n)}(X)$ consists of $6^n$ terms like $(-1)^{s_1 s_2} M_1^{r_1} M_2^{r_2} C_1^{s_1} C_2^{s_2} D_1^{t_1} X N_2^{m_2} m_2 C_2^{s_2} D_2^{t_2},$ where $s_1 + t_1 + s_2 + t_2 \geq n$.

Since $C_1, C_2, D_1, D_2$ are all nilpotents, we have $n_0$ such that $C_1^{n_0} = D_1^{n_0} = C_2^{n_0} = D_2^{n_0} = 0$. Thus for each term of $S^{(4n_0+1)}(X)$, as $s_1 + t_1 + s_2 + t_2 \geq 4n_0 + 1,$ we have at least one integer among $s_1, s_2, t_1, t_2$ greater than $n_0,$ so every term of $S^{(4n_0+1)}(X)$ is 0. Therefore, $S^{(4n_0+1)}(X) = 0.$ But

$$\Delta_{((N_1, N_2), (M_1, M_2))}^{(4n_0+1)}(X) = S^{(4n_0+1)}(X) = 0,$$

and $((N_1, N_2), (M_1, M_2))$ has the SPF theorem; so it follows that

$$\Delta_{((N_1, N_2), (M_1, M_2))}(X) = 0,$$  

or $N_1 XN_2 = M_1 XM_2.$

By Theorems 3.3 and 3.4, it is easy to see the following.

**Theorem 3.5.** Let $N, M \in B(H)$ and let $C, D$ be nilpotents such that $N, M, C, D$ mutually commute. If $M$ is invertible and $\|N\| M^{-1} \| < 1$, then $(N + C)X(N + C) = (M + D) X (M + D)$ implies $N X N = M X M$.

Moreover, if the strict inequality in Theorem 3.5 holds, then Theorem 3.5 is true even for the quasinilpotent operators.

**Theorem 3.6.** Let $N, M \in B(H)$ and let $C, D$ be quasinilpotents such that $N, M, C, D$ mutually commute. If $M$ is invertible and $\|N\| M^{-1} \| < 1$, then $(N + C)X(N + C) = (M + D) X (M + D)$ implies $X = 0$.

**Proof.** If $D$ is quasinilpotent and $M$ is invertible, then $M + D$ is invertible. If $(N + C)X(N + C) = (M + D) X (M + D)$ for some $X \in B(H)$, then

$$(N + C) (M + D)^{-1} X (N + C) (M + D)^{-1} = X$$

or

$$(NM^{-1} + F) X (NM^{-1} + F) = X,$$
where $F$ is quasinilpotent. By [3],

$$\sigma(\Delta_{(NM^{-1} + F, NM^{-1} + F), (I,I)}) = \sigma(NM^{-1})\sigma(NM^{-1}) - 1. \quad (3.20)$$

Since $\|N\|\|M^{-1}\| < 1$, $0$ is not in

$$\sigma(\Delta_{(NM^{-1} + F, NM^{-1} + F), (I,I)}), \quad (3.21)$$

and therefore $\Delta_{(NM^{-1} + F, NM^{-1} + F), (I,I)}$ is invertible. It follows from the equation

$$\Delta_{(NM^{-1} + F, NM^{-1} + F), (I,I)}(X) = 0 \quad (3.22)$$

that $X = 0$. 

The following results show that even if $((A, A), (B, B))$ has the SPF theorem, we still do not know if $((A^2, A^2), (B^2, B^2))$ has the SPF theorem.

**Theorem 3.7.** Let $A, B \in B(H)$. Let $\omega$ be an $n$th root of $1$, but $\omega^k \neq 1$ for $k$ such that $1 \leq k \leq n - 1$. If for any $k$ such that $0 \leq k \leq n - 1$, $((A, A), (B, \omega^k B))$ has the SPF theorem, then $((A^n, A^n), (B^n, B^n))$ has the SPF theorem too.

**Proof.** By induction, we can prove that

$$\Delta_{((A^n, A^n), (B^n, B^n))}(X) = \Delta_{((A, A), (B, B))}(\Delta_{((A, A), (B, \omega B))}(\cdots (\Delta_{((A, A), (B, \omega^{(n-1)} B))}(X) \cdots)). \quad (3.23)$$

Now if

$$\Delta^{(2)}_{((A^n, A^n), (B^n, B^n))}(X) = 0, \quad (3.24)$$

then

$$\Delta^{(2)}_{((A, A), (B, B))}(\Delta^{(2)}_{((A, A), (B, \omega B))}(\cdots (\Delta^{(2)}_{((A, A), (B, \omega^{(n-1)} B))}(X) \cdots)) = 0. \quad (3.25)$$

Since $((A, A), (B, B))$ has the SPF theorem, it follows that

$$\Delta_{((A, A), (B, B))}(\Delta^{(2)}_{((A, A), (B, \omega B))}(\cdots (\Delta^{(2)}_{((A, A), (B, \omega^{(n-1)} B))}(X) \cdots)) = 0. \quad (3.26)$$

or

$$\Delta^{(2)}_{((A, A), (B, \omega B))}(\Delta_{((A, A), (B, B))}(\cdots (\Delta^{(2)}_{((A, A), (B, \omega^{(n-1)} B))}(X) \cdots)) = 0, \quad (3.27)$$

and therefore

$$\Delta_{((A, A), (B, \omega B))}(\Delta_{((A, A), (B, B))}(\cdots (\Delta^{(2)}_{((A, A), (B, \omega^{(n-1)} B))}(X) \cdots)) = 0. \quad (3.28)$$

Proceeding in this way, we have finally

$$\Delta_{((A, A), (B, B))}(\Delta_{((A, A), (B, \omega B))}(\cdots (\Delta_{((A, A), (B, \omega^{(n-1)} B))}(X) \cdots)) = 0, \quad (3.29)$$
that is, by (3.23),
\[ \Delta_{((A^n, A^n), (B^n, B^n))}(X) = 0. \] (3.30)

The following result says that the converse of Theorem 3.8 is also true.

**Theorem 3.8.** Let \( A, B \in B(H) \). Let \( \omega \) be an \( n \)th root of 1, but \( \omega^k \neq 1 \) for \( k \) such that \( 1 \leq k \leq n - 1 \). If \( A \) or \( B \) is invertible and \((A^n, A^n), (B^n, B^n)\) has the SPF theorem, then for any \( k \) such that \( 0 \leq k \leq n - 1 \), \((A^n, (B, \omega^k B))\) has the SPF theorem too.

**Proof.** It is sufficient to prove that if \((A^n, B^n)\) has the SPF theorem and \( B \) is invertible, then \((A^n, (B, B))\) has the SPF theorem. Now if
\[ A(AXA - BXB)A = B(AXA - BXB)B, \] (3.31)
then
\[ A^n(AXA - BXB)A^n = B^n(AXA - BXB)B^n \] (3.32)
or
\[ A^n(AXA^n - B^nXB^n)A^n = B^n(A^nXA^n - B^nXB^n)B^n; \] (3.33)
so (3.24) holds. Since \((A^n, A^n), (B^n, B^n)\) has the SPF theorem, we have (3.30). It follows from (3.23) that (3.29) holds. From (3.31), we see that
\[ \Delta_{((A^n, A^n), (B^n, B^n))}((\Delta_{((A^n, A^n), (B^n, B^n))}(X)) \cdots) = 0. \] (3.34)

Note that
\[ \Delta_{((A^n, A^n), (B^n, B^n))}(Y) - \Delta_{((A^n, A^n), (B^n, B^n))}(Y) = (\omega - 1)BYB. \] (3.35)

Since \( B \) is invertible, (3.29) and (3.34) will give
\[ \Delta_{((A^n, A^n), (B^n, B^n))}((\Delta_{((A^n, A^n), (B^n, B^n))}(X)) \cdots) = 0. \] (3.36)

From (3.31), we see also that
\[ \Delta_{((A^n, A^n), (B^n, B^n))}((\Delta_{((A^n, A^n), (B^n, B^n))}(X)) \cdots) = 0; \] (3.37)
then (3.36) and (3.37) yields
\[ \Delta_{((A^n, A^n), (B^n, B^n))}((\Delta_{((A^n, A^n), (B^n, B^n))}(X)) \cdots) = 0. \] (3.38)

Proceeding in this way, we have finally
\[ \Delta_{((A^n, A^n), (B^n, B^n))}((\Delta_{((A^n, A^n), (B^n, B^n))}(X)) \cdots) = 0. \] (3.39)

Now (3.31) and (3.39) will give the desired equation: \( AXA - BXB = 0. \)
**Theorem 3.9.** If $C, D$ are nilpotents such that $CD = DC$ but $C^2 \neq D^2$, then $((C,C), (D,D))$ does not have the SPF theorem.

**Proof.** It is not difficult to see that

$$\Delta^{(n)}_{((C,C),(D,D))}(I) = \sum_{k=0}^{n} (-1)^k C^k C^{2n-2k} D^{2k},$$

(3.40)

where $I$ is the identity operator.

If $C, D$ are nilpotents, then there exists an $n_0$ such that $C^{n_0} = 0, D^{n_0} = 0$. For any $k$ such that $1 \leq k \leq n_0$, at least one of $2n_0 + 2 - 2k$ and $2k$ is greater than $n_0$. So by (3.40), we have

$$\Delta^{(n_0+1)}_{((C,C),(D,D))}(I) = 0.$$  

(3.41)

But $\Delta_{((C,C),(D,D))}(I) = C^2 - B^2 \neq 0$. This completes the proof.

\[\square\]

4. Asymptotic PF theorem and compact operators. We now give a theorem about the compact operators, which generalizes the relative result in [2].

**Theorem 4.1.** Let $A = (N_1,N_2)$ and $B = (M_1,M_2)$ be tuples of commuting normal operators in $B(H)$. If $X \in B(H)$ such that $\Delta^{(n)}_{(A,B)}(X)$ is compact for some $n \geq 2$, then $\Delta_{(A,B)}(X)$ is compact too.

**Proof.** Let $K(H)$ be the ideal of $B(H)$ consisting of all compact operators on $H$, let $B(H)/K(H)$ be the Calkin algebra, and let $\pi$ be the Calkin map from $B(H)$ to $B(H)/K(H)$. It is clear that

$$\pi\left(\Delta^{(n)}_{((N_1,N_2),(M_1,M_2))}(X)\right) = \Delta^{(n)}_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}(\pi(X)).$$

(4.1)

If $\Delta^{(n)}_{((N_1,N_2),(M_1,M_2))}(X)$ is compact, then $\pi\left(\Delta^{(n)}_{((N_1,N_2),(M_1,M_2))}(X)\right) = 0$. It follows that

$$\Delta^{(n)}_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}(\pi(X)) = 0.$$  

(4.2)

Since $\pi(N_i), \pi(M_i)$ are normal, for $i = 1,2$, applying Theorem 3.1, we have

$$\Delta_{((\pi(N_1),\pi(N_2)),(\pi(M_1),\pi(M_2)))}(\pi(X)) = 0.$$  

(4.3)

Therefore, $\Delta_{((N_1,N_2),(M_1,M_2))}(X)$ is compact. \[\square\]

The following theorem is an asymptotic version of the SPF theorem. It generalizes the corresponding result in [10].

**Theorem 4.2.** Let $A = (N_1,N_2)$ and $B = (M_1,M_2)$ be tuples of commuting normal operators in $B(H)$. Let $K$ be any positive real number and let $n$ be an integer greater than 1. Then for every neighborhood $U$ of 0 in $B(H)$ (under uniform, strong or weak topology), a neighborhood $V$ of 0 under the same topology is obtained such that if $\Delta^{(n)}_{(A,B)}(X) \in V$ and $\|X\| \leq K$, then $\Delta_{(A,B)}(X) \in U$. 


**Proof.** We first consider the following particular case: \( N_1 = N_2 = N, M_1 = M_2 = M \). Assume that \( \|N\| \) and \( \|M\| \) are not greater than 1 (if not, we can replace \( N \) and \( M \) by \( N/(\|N\| + \|M\|) \) and \( M/(\|N\| + \|M\|) \), resp.).

Let \( K > 0 \) and let \( U \) be any neighborhood of 0 in \( B(H) \) under uniform (or strong or weak) topology. Let \( U_{ij}, i, j = 1, 2, 3, 4 \), be neighborhoods of 0 in \( B(H) \) under the same topology such that

\[
\sum_{i=1}^{4} \sum_{j=1}^{4} U_{ij} \subset U. \tag{4.4}
\]

Suppose that \( N, M \) have the following spectral decomposition:

\[
N = \int_{\sigma(N)} \lambda dE_{\lambda}, \quad M = \int_{\sigma(M)} \lambda dF_{\lambda}. \tag{4.5}
\]

For any \( \epsilon > 0 \), define \( \Delta_{\epsilon} = \{ z \mid |z| \leq \epsilon \} \), \( \Delta_{\epsilon} = C \setminus \Delta_{\epsilon} \), and

\[
\begin{align*}
H_1(\epsilon) &= E(\Delta_{\epsilon})F(\Delta_{\epsilon})H, \\
H_2(\epsilon) &= E(\Delta_{\epsilon})F(\Delta_{\epsilon}^c)H, \\
H_3(\epsilon) &= E(\Delta_{\epsilon})F(\Delta_{\epsilon})H, \\
H_4(\epsilon) &= E(\Delta_{\epsilon})F(\Delta_{\epsilon}^c)H. \tag{4.6}
\end{align*}
\]

Then \( H \) can be written as \( H = H_1(\epsilon) \oplus H_2(\epsilon) \oplus H_3(\epsilon) \oplus H_4(\epsilon) \). Under this decomposition, we have

\[
N = \begin{pmatrix}
N_1(\epsilon) & & & \\
& N_2(\epsilon) & & \\
& & N_3(\epsilon) & \\
& & & N_4(\epsilon)
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
M_1(\epsilon) & & & \\
& M_2(\epsilon) & & \\
& & M_3(\epsilon) & \\
& & & M_4(\epsilon)
\end{pmatrix}, \tag{4.7}
\]

where \( \|N_1(\epsilon)\|, \|N_2(\epsilon)\|, \|M_1(\epsilon)\|, \|M_3(\epsilon)\| \) are not greater than \( \epsilon \), and \( N_3(\epsilon), N_4(\epsilon), M_2(\epsilon), \) and \( M_4(\epsilon) \) are invertible.

Let \( X = ((X_{ij}(\epsilon)))_{i,j=1,2,3,4} \) and let \( Z \) denote the set

\[
Z = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2),(4,1)\}. \tag{4.8}
\]

If \((k,l) \in Z\), then at least one operator in each pair of \((N_k,N_l), (M_k,M_l)\) has norm less
than $\epsilon$. Hence

$$\|N_k(\epsilon)X_{kl}(\epsilon)N_l(\epsilon) - M_k(\epsilon)X_{kl}(\epsilon)M_l(\epsilon)\| \to 0 \quad \text{as} \quad \epsilon \to 0.$$  \hspace{1cm} (4.9)

Therefore, we are able to choose a fixed number $\epsilon_0 > 0$ such that for each pair $(k, l) \in Z$,

$$(\delta_{ij}(k, l)\Delta_{((N_i(\epsilon_0),N_j(\epsilon_0)),(M_i(\epsilon_0),M_j(\epsilon_0)))}(X_{ij}(\epsilon_0)))_{4 \times 4} \in U_{kl},$$  \hspace{1cm} (4.10)

where $\delta_{ij}(k, l)$ equals 1 if $i = k$, $j = l$ and 0 otherwise. Set $V_{kl} = U_{kl}$.

For the sake of simplicity, we will omit $\epsilon_0$ in the notations of each component in the decompositions of $H$, $N$, $M$, $X$.

It is easy to see that $\Delta_{(A,B)}^{(n)}(X)$ has the following decomposition:

$$\Delta_{((N,N),(M,M))}^{(n)}(X) = (\Delta_{((N_i,N_j),(M_i,M_j))}^{(n)}(X_{ij}))_{4 \times 4}.$$  \hspace{1cm} (4.11)

If $(k, l)$ is not in $Z$, then at least one pair of $(N_k, N_l)$ and $(M_k, M_l)$ has two invertible operators. We assume that $N_k$ and $N_l$ are invertible (we can follow the same way if $M_k$, $M_l$ are invertible).

Let

$$O_{kl} = \{o_{kl} : (\delta_{ij}(k, l)o_{ij})_{4 \times 4} \in U_{ij}\}.$$  \hspace{1cm} (4.12)

Then $O_{kl}$ is a neighborhood of 0 in $B(H_k, H_k)$. Since $N_k$, $N_l$ are invertible, we can see that

$$\Delta_{((N_k,N_l),(M_k,M_l))}^{(n)}(X_{kl}) = N_{kl}^{n}\Delta_{((I_k,I_l),(N_{kl}^{-1},M_{kl}^{-1}))}^{(n)}(X_{kl})N_{kl}^{-n},$$  \hspace{1cm} (4.13)

where $I_k$, $I_l$ are identities on $H_k$, $H_l$. It follows from the asymptotic PF theorem in [2] that there is the neighborhood $P_{kl}$ of 0 in $B(H_k, H_k)$ such that for $\|X_{kl}\| \leq K$, if

$$\Delta_{((I_k,I_l),(N_{kl}^{-1},M_{kl}^{-1}))}^{(n)}(X_{kl}) \in P_{kl},$$  \hspace{1cm} (4.14)

then

$$\Delta_{((I_k,I_l),(N_{kl}^{-1},M_{kl}^{-1}))}^{(n)}(X_{kl}) \in N_{kl}^{-1}O_{kl}N_{kl}^{-1}.$$  \hspace{1cm} (4.15)

Set $V_{kl} = N_{kl}^{-n}P_{kl}N_{kl}^{n}$. If

$$\Delta_{((N_k,N_l),(M_k,M_l))}^{(n)}(X_{kl}) \in V_{kl},$$  \hspace{1cm} (4.16)

then

$$\Delta_{((N_k,N_l),(M_k,M_l))}^{(n)}(X_{kl}) \in O_{kl}.$$  \hspace{1cm} (4.17)

Let

$$V = \{(u_{ij})_{4 \times 4} : \forall u_{ij} \in V_{ij}\}.$$  \hspace{1cm} (4.18)
Then \( V \) is a neighborhood of 0. If \( \|X\| \leq K \) and \( \Delta_{(A,B)}^{(n)}(X) \in V \), then for each pair \((k,l)\), \( \|X_{kl}\| \leq K \) and (4.16) holds; so it follows that (4.17) holds, that is,

\[
(\delta_{ij}(k,l)\Delta_{((N_k,N_l),(M_k,M_l))}(X_{kl}))_{4 \times 4} \in U_{kl},
\]

but

\[
\Delta_{(A,B)}(X) = \sum_{k=1}^{4} \sum_{l=1}^{4} (\delta_{ij}(k,l)\Delta_{((N_k,N_l),(M_k,M_l))}(X_{kl}))_{4 \times 4},
\]

which is in \( U \) by (4.4).

In general, let

\[
\tilde{N} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.
\]

Then \( \tilde{N}, \tilde{M} \) are normal in \( B(H \oplus H) \). Let

\[
\tilde{U} = \left\{ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} : u_i \in U, \ i = 1, 2, 3, 4 \right\}.
\]

\( \tilde{U} \) is a neighborhood of 0 in \( B(H \oplus H) \). So there is a neighborhood \( \tilde{V} \) of 0 in \( B(H \oplus H) \) such that if \( \|\tilde{X}\| \leq K, \Delta_{(A,B)}^{(n)}(\tilde{X}) \in \tilde{V} \), then \( \Delta_{(A,B)}(\tilde{X}) \in \tilde{U} \). Let

\[
\tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad \tilde{V} = \left\{ v : \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \right\}.
\]

\( \tilde{V} \) is a neighborhood of 0 in \( B(H) \). If \( \|X\| \leq K, \Delta_{(A,B)}^{(n)}(X) \in V \), then \( \|\tilde{X}\| \leq K \) and \( \Delta_{(A,B)}^{(n)}(\tilde{X}) \in \tilde{V} \); so \( \Delta_{(A,B)}(\tilde{X}) \in \tilde{U} \), which means that

\[
\begin{pmatrix} 0 & \Delta_{(A,B)}(X) \\ 0 & 0 \end{pmatrix} \in \tilde{U}
\]

or \( \Delta_{(A,B)}(X) \in U \).

Using the same technique, we are able to generalize the asymptotic PF theorems obtained by Moore [6] and Rogers [8].

**Theorem 4.3.** Let \( N_1, N_2, M_1, M_2, k \) be the same as in Theorem 4.2. Then for any neighborhood \( U \) of 0 in \( B(H) \) (under uniform, strong or weak topology), a neighborhood \( V \) of 0 under the same topology is obtained such that if \( N_1^* X N_2^* - M_1^* X M_2^* \in V \) and \( \|X\| \leq K \), then \( N X N - M X M \in U \).

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