COMMON FIXED POINT THEOREMS OF
CONTRACTIVE-TYPE MAPPINGS

HEE SOO PARK and JEONG SHEOK UME

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Using the concept of $D$-metric we prove some common fixed point theorems for generalized contractive mappings on a complete $D$-metric space. Our results extend, improve, and unify results of Fisher and Ćirić.

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1. Introduction. The Banach contraction mapping principle is well known. There are many generalizations of that principle to single- and multivalued mappings (see [1, 4, 5, 10, 11, 12]). The study of maps satisfying some contractive conditions has been the center of rigorous research activity since such mappings have many applications (see [2, 3, 9, 13, 14, 15]).

In 1998, Ćirić [6] proved a common fixed point theorem for nonlinear mappings on a complete metric space: let $(X, d)$ be a complete metric space and $S, T : X \to X$ self-maps such that $d(STx, TSy) \leq \max\{\varphi_1[(1/2)(d(x, Sy) + d(y, Tx))], \varphi_2[d(x, Tx)], \varphi_3[d(y, Sy)], \varphi_4[d(x, y)]\}$ for all $x, y \in X$, where $\varphi_i \in \Phi$ ($i = 1, 2, 3, 4$). If $S$ or $T$ is continuous, then $S$ and $T$ have a unique common fixed point. This result improved and extended a theorem of Fisher [8].

In this paper, using the concept of $D$-metric, we prove common fixed point theorems which extend, improve, and unify the corresponding theorems of Fisher [8] and Ćirić [6].

Throughout the paper, by $\Phi$ we denote the collection of functions $\varphi : [0, \infty) \to [0, \infty)$ which are continuous from the right, nondecreasing, and which satisfy the condition $\varphi(t) < t$ for all $t > 0$. We denote by $\mathbb{N}$ the set of all positive integers.

2. Preliminaries. Before proving the main theorem, we will introduce some definitions and lemmas.

**Definition 2.1** [7]. Let $X$ be any nonempty set. A $D$-metric for $X$ is a function $D : X \times X \times X \to \mathbb{R}$ such that

1. $D(x, y, z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x = y = z$,
2. $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$ for all $x, y, z \in X$,
3. $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z \in X$.

If $D$ is a $D$-metric for $X$, then the ordered pair $(X, D)$ is called a $D$-metric space or the set $X$, together with a $D$-metric, is called a $D$-metric space. We note that to
a given ordinary metric space \((X,d)\) there corresponds a \(D\)-metric space \((X,D)\), but the converse may not be true (see Example 3.3). In this sense the \(D\)-metric spaces are the generalizations of the ordinary metric space.

**Definition 2.2** [7]. A sequence \(\{x_n\}\) of points of a \(D\)-metric space \(X\) converges to a point \(x \in X\) if for an arbitrary \(\varepsilon > 0\), there exists an \(n_0 \in \mathbb{N}\) such that for all \(n > m \geq n_0\), 
\[
D(x_m, x_n, x) < \varepsilon.
\]

**Definition 2.3** [7]. A sequence \(\{x_n\}\) of points of a \(D\)-metric space \(X\) is said to be a \(D\)-Cauchy sequence if for an arbitrary \(\varepsilon > 0\), there exists an \(n_0 \in \mathbb{N}\) such that for all \(p > n > m \geq n_0\), 
\[
D(x_m, x_n, x_p) < \varepsilon.
\]

**Definition 2.4** [7]. A \(D\)-metric space \(X\) is a complete \(D\)-metric space if every \(D\)-Cauchy sequence \(\{x_n\}\) in \(X\) converges to a point \(x\) in \(X\).

**Definition 2.5**. A real-valued function \(f\) defined on a metric space \(X\) is said to be lower semicontinuous at a point \(t\) in \(X\) if 
\[
\lim_{x \to t} \inf f(x) = \infty \quad \text{or} \quad \lim_{x \to t} \inf f(x) \geq f(t).
\]

**Definition 2.6**. A real-valued function \(f\) defined on a metric space \(X\) is said to be upper semicontinuous at a point \(t\) in \(X\) if 
\[
\lim_{x \to t} \sup f(x) = \infty \quad \text{or} \quad \lim_{x \to t} \sup f(x) \leq f(t).
\]

**Definition 2.7**. Let \(x_0 \in X\) and \(\varepsilon > 0\) be given. Then the open ball \(B(x_0, \varepsilon)\) in \(X\) centered at \(x_0\) of radius \(\varepsilon\) is defined by 
\[
B(x_0, \varepsilon) = \left\{ y \in X \mid D(x_0, y, y) < \varepsilon \text{ if } y = x_0, \sup_{z \in X} D(x_0, y, z) < \varepsilon \text{ if } y \neq x_0 \right\}.
\]

Then the collection of all open balls \(\{B(x, \varepsilon) : x \in X\}\) defines the topology on \(X\) denoted by \(\tau\).

**Lemma 2.8** [7]. The \(D\)-metric for \(X\) is a continuous function on \(X \times X \times X\) in the topology \(\tau\) on \(X\).

**Lemma 2.9** [6]. If \(\varphi_1, \varphi_2 \in \Phi\), then there is some \(\varphi \in \Phi\) such that 
\[
\max \{\varphi_1(t), \varphi_2(t)\} \leq \varphi(t) \text{ for all } t > 0.
\]

**Lemma 2.10**. Let \((X,D)\) be a \(D\)-metric space. Let \(g : X \times X \to X\) be a mapping and let \(S, T : X \to X\) be mappings such that 
\[
\max \left\{ D(STx, TSy, g(STx, TSy)), D(TSy, STx, g(TSy, STx)) \right\} 
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} \left( D(x, Sy, g(x, Sy)) + D(y, Tx, g(y, Tx)) \right) \right], \varphi_2 \left[ D(x, Tx, g(x, Tx)) \right], \varphi_3 \left[ D(y, Sy, g(y, Sy)) \right], \varphi_4 \left[ D(x, y, g(x, y)) \right] \right\},
\]
\[
\text{for all } x, y \in X, \text{ where } \varphi_i \in \Phi \text{ (} i = 1, 2, 3, 4\),
\]
\[
x = y \Rightarrow D(x, y, g(x, y)) = 0.
\]
and

$$\max \{D(x,z,g(x,z)),D(x,y,g(x,z)),D(y,z,g(x,z))\} \leq D(x,y,g(x,y)) + D(y,z,g(y,z))$$

(2.4)

for all \(x, y, z \in X\). The sequence \(\{x_n\}\) is defined by \(x_0 \in X\), \(x_{2n+1} = Tx_{2n}\), and \(x_{2n+2} = Sx_{2n+1}\) for every \(n \in \mathbb{N} \cup \{0\}\). Then

(i) for an arbitrary \(\varepsilon > 0\), there exists a positive integer \(L\) such that \(L \leq n < m\) implies

$$\max \{D(x_n,x_m,g(x_n,x_m)),D(x_m,x_n,g(x_m,x_n))\} < \varepsilon,$$

(ii) a sequence \(\{x_n\}_{n=0}^{\infty}\) is a \(D\)-Cauchy sequence.

**Proof.** Let \(M = \max \{D(x_0,x_1,g(x_0,x_1)),D(x_1,x_2,g(x_1,x_2)),D(x_2,x_3,g(x_2,x_3))\}\). Since all \(\varphi_i\) are nondecreasing functions by (2.2), (2.3), and (2.4),

$$\max \{D(x_2,x_3,g(x_2,x_3)),D(x_3,x_2,g(x_3,x_2))\}$$

$$= \max \{D(STx_0,TSx_1,g(STx_0,TSx_1)),D(TSx_1,STx_0,g(TSx_1,STx_0))\}$$

$$\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x_0,Sx_1,g(x_0,Sx_1)) + D(x_1,Tx_0,g(x_1,Tx_0))) \right], \varphi_2 [D(x_0,Tx_0,g(x_0,Tx_0))], \varphi_3 [D(x_1,Sx_1,g(x_1,Sx_1))], \varphi_4 [D(x_0,x_1,g(x_0,x_1))] \right\},$$

(2.5)

$$\leq \max \{\varphi_1(M),\varphi_2(M),\varphi_3(M),\varphi_4(M)\} \leq \varphi(M),$$

where \(\varphi \in \Phi\). Such \(\varphi\) exists from an extended version of Lemma 2.9. Therefore, we have \(\max \{D(x_2,x_3,g(x_2,x_3)),D(x_3,x_2,g(x_3,x_2))\} \leq \varphi(M)\). Again, from (2.2), (2.3), and (2.4), we get

$$\max \{D(x_3,x_4,g(x_3,x_4)),D(x_4,x_3,g(x_4,x_3))\}$$

$$= \max \{D(TSx_1,STx_2,g(TSx_1,STx_2)),D(STx_2,TSx_1,g(STx_2,TSx_1))\}$$

$$\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x_2,Sx_1,g(x_2,Sx_1)) + D(x_1,Tx_2,g(x_1,Tx_2))) \right], \varphi_2 [D(x_2,Tx_2,g(x_2,Tx_2))], \varphi_3 [D(x_1,Sx_1,g(x_1,Sx_1))], \varphi_4 [D(x_2,x_1,g(x_2,x_1))] \right\},$$

(2.6)

$$\leq \max \{\varphi_1(M),\varphi_2 [\varphi(M)],\varphi_3(M),\varphi_4(M)\} \leq \varphi(M).$$

Using the obtained relations \(\max \{D(x_2,x_3,g(x_2,x_3)),D(x_3,x_2,g(x_3,x_2))\} \leq \varphi(M)\) and \(\max \{D(x_3,x_4,g(x_3,x_4)),D(x_4,x_3,g(x_4,x_3))\} \leq \varphi(M)\), from (2.2), (2.3), and (2.4),
we get

\[
\max \{D(x_4, x_5, g(x_4, x_5)), D(x_5, x_4, g(x_5, x_4))\} = \max \{D(STx_2, TSx_3, g(STx_2, TSx_3)), D(TSx_3, STx_2, g(TSx_3, STx_2))\}
\]

\[
\leq \max \left\{ \phi_1 \left[ \frac{1}{2} (D(x_2, Sx_3, g(x_2, Sx_3)) + D(x_3, Tx_2, g(x_3, Tx_2))) \right], \phi_2 [D(x_2, Tx_2, g(x_2, Tx_2))] \right\}, \phi_3 [D(x_3, Sx_3, g(x_3, Sx_3))], \phi_4 [D(x_2, x_3, g(x_2, x_3))]
\]

\leq \max \{ \phi_1 [\phi(M)], \phi_2 [\phi(M)], \phi_3 [\phi(M)], \phi_4 [\phi(M)] \}
\]

\leq \phi^2 (M).

Similarly, again from (2.2), (2.3), and (2.4), we get

\[
\max \{D(x_5, x_6, g(x_5, x_6)), D(x_6, x_5, g(x_6, x_5))\} = \max \{D(TSx_3, STx_4, g(TSx_3, STx_4)), D(STx_4, TSx_3, g(STx_4, TSx_3))\}
\]

\[
\leq \max \left\{ \phi_1 \left[ \frac{1}{2} (D(x_4, Sx_3, g(x_4, Sx_3)) + D(x_3, Tx_4, g(x_3, Tx_4))) \right], \phi_2 [D(x_4, Tx_4, g(x_4, Tx_4))] \right\}, \phi_3 [D(x_3, Sx_3, g(x_3, Sx_3))], \phi_4 [D(x_4, x_3, g(x_4, x_3))]
\]

\leq \max \{ \phi_1 [\phi(M)], \phi_2 [\phi^2(M)], \phi_3 [\phi(M)], \phi_4 [\phi(M)] \}
\]

\leq \phi^2 (M).

In general, by induction, we get

\[
\max \{D(x_n, x_{n+1}, g(x_n, x_{n+1})), D(x_{n+1}, x_n, g(x_{n+1}, x_n))\} \leq \phi^{[n/2]} (M)
\]

for \( n \geq 2 \), where \([n/2]\) stands for the greatest integer not exceeding \( n/2 \). Since \( \phi \in \Phi \), by Singh and Meade [13, Lemma 1], it follows that \( \phi^n (M) \to 0 \) as \( n \to +\infty \) for every \( M > 0 \). Thus, we obtain

\[
\max \{D(x_n, x_{n+1}, g(x_n, x_{n+1})), D(x_{n+1}, x_n, g(x_{n+1}, x_n))\} \to 0 \quad \text{as} \quad n \to \infty.
\]

Suppose that (I) does not hold. Then there exists an \( \varepsilon > 0 \) such that for each \( i \in \mathbb{N} \), there exist positive integers \( n_i, m_i \), with \( i \leq n_i < m_i \), satisfying

\[
\varepsilon \leq \max \{D(x_{n_i}, x_{m_i}, g(x_{n_i}, x_{m_i})), D(x_{m_i}, x_{n_i}, g(x_{m_i}, x_{n_i}))\},
\]

\[
\max \{D(x_{n_i}, x_{m_i-1}, g(x_{n_i}, x_{m_i-1})), D(x_{m_i-1}, x_{n_i}, g(x_{m_i-1}, x_{n_i}))\} < \varepsilon \quad \text{for} \quad i = 1, 2, \ldots
\]

(2.11)

Set

\[
\varepsilon_i = \max \{D(x_{n_i}, x_{m_i}, g(x_{n_i}, x_{m_i})), D(x_{m_i}, x_{n_i}, g(x_{m_i}, x_{n_i}))\},
\]

\[
\rho_i = \max \{D(x_i, x_{i+1}, g(x_i, x_{i+1})), D(x_{i+1}, x_i, g(x_{i+1}, x_i))\} \quad \text{for} \quad i = 1, 2, \ldots
\]

(2.12)
Then we have
\[
\varepsilon \leq \varepsilon_i
= \max \{ D(x_n, x_{m_1}, g(x_{n_1}, x_{m_1})), D(x_{m_1}, x_{n_1}, g(x_{m_1}, x_{n_1})) \}
\leq \max \{ D(x_n, x_{m_1-1}, g(x_{n_1}, x_{m_1-1})), D(x_{m_1-1}, x_{n_1}, g(x_{m_1-1}, x_{n_1})) \}
+ \max \{ D(x_{m_1-1}, x_{m_1}, g(x_{m_1-1}, x_{m_1})), D(x_m, x_{m_1-1}, g(x_m, x_{m_1-1})) \}
< \varepsilon + \rho_{m_1-1}, \quad i = 1, 2, \ldots
\]
Taking the limit as \( i \to +\infty \), we get \( \lim \varepsilon_i = \varepsilon \). On the other hand, by (2.2), (2.3), and (2.4),
\[
\varepsilon_i = \max \{ D(x_n, x_{m_1}, g(x_{n_1}, x_{m_1})), D(x_{m_1}, x_{n_1}, g(x_{m_1}, x_{n_1})) \}
\leq \max \{ D(x_n, x_{n_1+1}, g(x_{n_1}, x_{n_1+1})), D(x_{n_1+1}, x_{n_1}, g(x_{n_1+1}, x_{n_1})) \}
+ \max \{ D(x_{n_1+2}, x_{n_1+2}, g(x_{n_1+2}, x_{n_1+2})), D(x_{n_1+2}, x_{n_1+1}, g(x_{n_1+2}, x_{n_1+1})) \}
+ \max \{ D(x_{n_1+2}, x_{m_1+2}, g(x_{n_1+2}, x_{m_1+2})), D(x_{m_1+2}, x_{n_1+2}, g(x_{m_1+2}, x_{n_1+2})) \}
+ \max \{ D(x_{m_1+1}, x_{m_1+1}, g(x_{m_1+1}, x_{m_1+1})), D(x_{m_1+1}, x_{m_1+2}, g(x_{m_1+1}, x_{m_1+2})) \}
\]
\[
= \rho_{n_i} + \rho_{n_i+1} + \max \{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})),
D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \}
+ \rho_{m_i+1} + \rho_{m_i}, \quad i = 1, 2, \ldots
\]

We will now analyze the term \( \max \{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \} \) based on the parity of the subscripts.

**Case 1.** \( n_i + 2 \) is even and \( m_i + 2 \) is odd. From (2.2), (2.3), and (2.4), we have
\[
\max \{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \}
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x_{n_i}, g(x_{n_i}, S_{n_i})), D(x_{m_i}, g(x_{m_i}, T_{n_i})) \right], \varphi_2 \left[ D(x_{n_i}, g(x_{n_i}, T_{n_i})), \varphi_3 \left[ D(x_{m_i}, g(x_{m_i}, S_{m_i})) \right], \varphi_4 \left[ D(x_{n_i}, g(x_{n_i}, x_{m_i})) \right] \right] \right\}
\leq \max \{ \varphi_1 \left[ \frac{1}{2} (\varepsilon_i + \rho_{m_i} + \varepsilon_i + \rho_{n_i}) \right], \varphi_2 (\rho_{n_i}), \varphi_3 (\rho_{m_i}), \varphi_4 (\varepsilon_i) \}
\leq \varphi (\varepsilon_i + \rho_{m_i} + \rho_{n_i}).
\]

Therefore, we have
\[
\max \{ D(x_{n_i+2}, x_{m_i+2}, g(x_{n_i+2}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \} \leq \varphi (k_i),
\]
where $k_i = \varepsilon_i + \rho_{m_i} + \rho_{n_i}$. Substituting (2.16) into (2.14), taking the limit as $i \to +\infty$, and using the right continuity of $\varphi$, we get

$$
\varepsilon = \lim_{i \to \infty} \varepsilon_i \leq \lim_{k_i \to \varepsilon^+} \varphi(k_i) = \varphi(\varepsilon) < \varepsilon,
$$

(2.17)

which is a contradiction.

**CASE 2.** Both $n_i + 2$ and $m_i + 2$ are odd. Then, we have

$$
\max \left\{ D(x_{n_i+1}, x_{m_i+1}, g(x_{n_i+1}, x_{m_i+1})), D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})) \right\}
\leq \max \left\{ D(x_{n_i+1}, x_{n_i+1}, g(x_{n_i+1}, x_{n_i+1})), D(x_{n_i+1}, x_{n_i+2}, g(x_{n_i+1}, x_{n_i+2})) \right\}
+ \max \left\{ D(x_{m_i+2}, x_{n_i+2}, g(x_{m_i+2}, x_{n_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \right\}
= \rho_{n_i+1} + \max \left\{ D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \right\}.
$$

(2.18)

Since $n_i + 1$ is even and $m_i + 2$ is odd, from Case 1, we have

$$
\max \left\{ D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \right\}
= \max \left\{ D(STx_{n_i-1}, TSx_{m_i}, g(STx_{n_i-1}, TSx_{m_i})), D(STx_{m_i}, STx_{n_i-1}, g(STx_{m_i}, STx_{n_i-1})) \right\}
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (D(x_{n_i-1}, Sx_{m_i}, g(x_{n_i-1}, Sx_{m_i}))) + D(x_{m_i}, Tx_{n_i-1}, g(x_{m_i}, Tx_{n_i-1})) \right], \varphi_2 [D(x_{n_i-1}, Tx_{n_i-1}, g(x_{n_i-1}, Tx_{n_i-1}))], \varphi_3 [D(x_{m_i}, Sx_{m_i}, g(x_{m_i}, Sx_{m_i}))], \varphi_4 [D(x_{n_i-1}, x_{m_i}, g(x_{n_i-1}, x_{m_i}))] \right\}
\leq \max \left\{ \varphi_1 \left[ \frac{1}{2} (\rho_{n_i-1} + \varepsilon_i + \rho_{m_i} + \varepsilon_i) \right], \varphi_2 (\rho_{n_i-1}), \varphi_3 (\rho_{m_i}), \varphi_4 (\rho_{n_i-1} + \varepsilon_i) \right\}
\leq \varphi (\varepsilon_i + \rho_{m_i} + \rho_{n_i-1}).
$$

(2.19)

Therefore, we get

$$
\max \left\{ D(x_{n_i+1}, x_{m_i+2}, g(x_{n_i+1}, x_{m_i+2})), D(x_{m_i+2}, x_{n_i+1}, g(x_{m_i+2}, x_{n_i+1})) \right\} \leq \varphi (l_i),
$$

(2.20)

where $l_i = \varepsilon_i + \rho_{m_i} + \rho_{n_i-1}$. Hence, substituting (2.20) into (2.18), then putting (2.18) into (2.14), and taking the limit as $i \to +\infty$, we have

$$
\varepsilon = \lim_{i \to \infty} \varepsilon_i \leq \lim_{l_i \to \varepsilon^+} \varphi(l_i) = \varphi(\varepsilon) < \varepsilon,
$$

(2.21)

which is a contradiction. In a similar manner, we get (2.17) and (2.21) for the cases in which $n_i + 2$ and $m_i + 2$ are both even, and $n_i + 2$ is odd and $m_i + 2$ is even. That is, all cases lead to a contradiction. Therefore (I) holds.
We claim that \( \{x_n\} \) is \( D \)-Cauchy. Let \( n, m, p \) \( (n < m < p) \) be any positive integers. Then, by Definition 2.1 and (2.4),

\[
D(x_n, x_m, x_p) \leq D(x_n, x_m, g(x_n, x_m)) + D(x_n, x_p, g(x_n, x_m)) + D(x_m, x_p, g(x_n, x_m)) \\
\leq D(x_n, x_m, g(x_n, x_m)) + 2D(x_n, x_m, g(x_n, x_m)) + 2D(x_m, x_p, g(x_n, x_m)) \\
= 3D(x_n, x_m, g(x_n, x_m)) + 2D(x_m, x_p, g(x_n, x_m)).
\]

(2.22)

Since \( \lim_{n \to \infty} D(x_n, x_m, g(x_n, x_m)) = 0 \), we have \( \lim_{n \to \infty} D(x_n, x_m, x_p) = 0 \). Thus \( \{x_n\} \) is a \( D \)-Cauchy sequence.

3. Main results. Now we will prove the following fixed point theorems for a complete \( D \)-metric space.

**Theorem 3.1.** Let \((X, D)\) be a complete \( D \)-metric space. Let \( g : X \times X \to X \) be a function and let \( S \) and \( T \) be self-maps on \( X \) satisfying (2.2), (2.3), and (2.4) of Lemma 2.10. For any sequences \( \{u_n\}, \{v_n\} \) in \( X \) such that \( \lim_{n \to \infty} u_n = \alpha \) and \( \lim_{n \to \infty} v_n = \beta \),

\[
\lim_{n \to \infty} D(u_n, v_n, g(u_n, v_n)) = D(\alpha, \beta, g(\alpha, \beta))
\]

for some \( \alpha, \beta \) in \( X \).

If \( S \) or \( T \) is continuous, then \( S \) and \( T \) have a unique common fixed point.

**Proof.** Let the sequence \( \{x_n\} \) be defined by \( x_0 \in X \), \( x_{2n+1} = Tx_{2n} \), and \( x_{2n+2} = Sx_{2n+1} \) for every \( n \in \mathbb{N} \cup \{0\} \). Then, by Lemma 2.10(II), it follows that \( \{x_n\} \) is a \( D \)-Cauchy sequence. Since \( X \) is a complete \( D \)-metric space, \( \{x_n\} \) is convergent to a limit \( u \) in \( X \). Suppose that \( S \) is continuous. Then

\[
u = \lim_{n \to \infty} x_{2n+2} = \lim_{n \to \infty} Sx_{2n+1} = S \left( \lim_{n \to \infty} x_{2n+1} \right) = Su.
\]

(3.1)

This implies that \( u \) is a fixed point of \( S \). From (2.2), (2.3), and (2.4), we get

\[
D(u, Tu, g(u, Tu)) = D(u, Tsu, g(u, Tsu)) \\
\leq D(u, x_{2n+2}, g(u, x_{2n+2})) + D(STx_{2n}, Tsu, g(STx_{2n}, Tsu)) \\
\leq D(u, x_{2n+2}, g(u, x_{2n+2})) + \max \{ \varphi_1 \left[ \frac{1}{2} D(x_{2n}, Su, g(x_{2n}, Su)) + D(u, Tx_{2n}, g(u, Tx_{2n})) \right], \\
\varphi_2 [D(x_{2n}, Tx_{2n}, g(x_{2n}, Tx_{2n}))], \varphi_3 [D(u, Su, g(u, Su))], \\
\varphi_4 [D(x_{2n}, u, g(x_{2n}, u))] \}.
\]

(3.2)

Taking the limit when \( n \) tends to infinity, by hypothesis, we get \( D(u, Tu, g(u, Tu)) = 0 \). Thus, we have \( u = Su = Tu \). Therefore, \( u \) is the common fixed point of \( S \) and \( T \). The proof for \( T \) continuous is similar.
We will now show that $u$ is unique. Suppose that $v$ is also a common fixed point of $S$ and $T$. Then, from (2.2), (2.3), and (2.4),

$$\max \{D(u, v, g(u, v)), D(v, u, g(v, u))\}$$

$$= \max \{D(STu, TSv, g(STu, TSv)), D(TSv, STu, g(TSv, STu))\}$$

$$\leq \max \left\{ \varphi_1 \left( \frac{1}{2} (D(u, Sv, g(u, Sv)) + D(v, Tu, g(v, Tu))) \right), \varphi_2 [D(u, Tu, g(u, Tu))], \varphi_3 [D(v, Sv, g(v, Sv))], \varphi_4 [D(u, v, g(u, v))] \right\}$$

$$= \max \left\{ \varphi_1 \left( \frac{1}{2} (D(u, v, g(u, v)) + D(v, u, g(v, u))) \right), \varphi_2 [D(u, u, g(u, u))], \varphi_3 [D(v, v, g(v, v))], \varphi_4 [D(u, v, g(u, v))] \right\}$$

$$\leq \varphi(\max \{D(u, v, g(u, v)), D(v, u, g(v, u))\})$$

(3.3)

We write $\max \{D(u, v, g(u, v)), D(v, u, g(v, u))\} \leq \varphi(\max \{D(u, v, g(u, v)), D(v, u, g(v, u))\})$, which implies that $\max \{D(u, v, g(u, v)), D(v, u, g(v, u))\} = 0$, that is, $u = v$. Therefore, the common fixed point of $S$ and $T$ is unique.

**Remark 3.2.** Let $X$ be a complete metric space with a metric $d$. If we take $D(x, y, z) = \max \{d(x, y), d(x, z), d(y, z)\}$ and $g(x, y) = x$ for all $x, y, z \in X$, then Theorem 3.1 is Ćirić’s [6, Theorem 2] which has extended a theorem of Fisher [8].

The following example shows that a $D$-metric is a proper extension of a metric $d$.

**Example 3.3.** Let $d$ be a metric on $\mathbb{R}$. Define the function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\varphi(x, y) = (x - y)^2$ for all $x, y \in \mathbb{R}$. Then, clearly, $\varphi$ is not metric since $\varphi(2, 1/2) > \varphi(2, 1) + \varphi(1, 1/2)$. Let $G, H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be functions such that $G(x, y, z) = \max \{d(x, y), d(x, z), d(y, z)\}$ and $H(x, y, z) = \max \{\varphi(x, y), \varphi(x, z), \varphi(y, z)\}$ for all $x, y, z \in \mathbb{R}$. Then, clearly, $G$ and $H$ are $D$-metric for $\mathbb{R}$. But $H$ is a $D$-metric that is a proper extension of the metric $d$. Therefore, a $D$-metric space is a proper extension of a metric space.

**Corollary 3.4.** Let $(X, D)$ be a complete $D$-metric space. Let $g : X \times X \to X$ be a function and let $S$ and $T$ be self-maps on $X$ satisfying

$$\max \{D(STx, TSy, g(STx, TSy)), D(TSy, STx, g(TSy, STx))\}$$

$$\leq c \cdot \max \left\{ \frac{1}{2} \left[ D(x, Sy, g(x, Sy)) + D(y, Tx, g(y, Tx)) \right], D(x, Tx, g(x, Tx)), D(y, Sy, g(y, Sy)), D(x, y, g(x, y)) \right\}$$

(3.4)

for all $x, y \in X$, where $x = y$ implies $D(x, y, g(x, y)) = 0$ and $\max \{D(x, z, g(x, z)), D(x, y, g(x, z)), D(y, z, g(x, z))\} \leq D(x, y, g(x, y)) + D(y, z, g(y, z))$ for all $x, y, z \in X$. 
For any sequences \( \{u_n\}, \{v_n\} \) in \( X \) such that \( \lim_{n \to \infty} u_n = \alpha \) and \( \lim_{n \to \infty} v_n = \beta \), \( \lim_{n \to \infty} D(u_n, v_n, g(u_n, v_n)) = D(\alpha, \beta, g(\alpha, \beta)) \) for some \( \alpha, \beta \) in \( X \).

If \( S \) or \( T \) is continuous, then \( S \) and \( T \) have a unique common fixed point.

**Proof.** The proof follows by taking \( \phi_i(t) = c \cdot t \) with \( 0 < c < 1 \) \( (i = 1,2,3,4) \) in Theorem 3.1.

We will prove the following corollary using another condition instead of continuity in Theorem 3.1.

**Corollary 3.5.** Let \( (X,D) \) be a complete \( D \)-metric space. Let \( g : X \times X \to X \) be a function, let \( S \) and \( T \) be self-maps on \( X \) satisfying (2.2), (2.3), and (2.4) of Lemma 2.10, and, for each \( u \in X \) with \( u \neq Su \) or \( u \neq Tu \), let

\[
\inf \{D(x,u,g(x,u)) + D(x,Sx, g(x,Sx)) + D(y,Ty, g(y,Ty)) : x, y \in X\} > 0.
\]

(3.5)

For any sequences \( \{a_n\} \) and \( \{b_n\} \) in \( X \) such that \( \lim_{n \to \infty} a_n = u \) and \( \lim_{n \to \infty} b_n = v \), the following conditions hold:

1. \( \lim_{n \to \infty} D(a_n, b_n, g(a_n, b_n)) = D(u, v, g(u, v)) \),
2. \( \lim_{n \to \infty} D(a_n, b_m, g(a_n, b_m)) = D(a_n, v, g(a_n, v)) \) for each \( n \in \mathbb{N} \),
3. \( \lim_{n \to \infty} D(b_m, a_n, g(b_m, a_n)) = D(v, a_n, g(v, a_n)) \) for each \( n \in \mathbb{N} \).

Then \( S \) and \( T \) have a unique common fixed point.

**Proof.** From Lemma 2.10(i) and (ii), the sequence \( \{x_n\} \) defined by \( x_0 \in X, x_{2n+1} = Tx_{2n}, \) and \( x_{2n+2} = Sx_{2n+1} \) for every \( x \in \mathbb{N} \cup \{0\} \) is a \( D \)-Cauchy sequence. Since \( X \) is a complete \( D \)-metric space, there exists \( u \in X \) such that \( \{x_n\} \) converges to \( u \). Then we have

\[
D(x_{2n+1}, x_{2m+2}, g(x_{2n+1}, x_{2m+2}))
\]

\[
= D(TSx_{2n-1}, STx_{2m}, g(TSx_{2n-1}, STx_{2m}))
\]

\[
\leq \max \left\{ \phi_1 \left[ \frac{1}{2} D(x_{2m}, Sx_{2n-1}, g(x_{2m}, Sx_{2n-1})) + D(x_{2n-1}, Tx_{2m}, g(x_{2n-1}, Tx_{2m})) \right] \right\},
\]

\[
\phi_2 \left[ D(x_{2m}, Tx_{2m}, g(x_{2m}, Tx_{2m})) \right], \phi_3 \left[ D(x_{2n-1}, Sx_{2n-1}, g(x_{2n-1}, Sx_{2n-1})) \right],
\]

\[
\phi_4 \left[ D(x_{2m}, x_{2n-1}, g(x_{2m}, x_{2n-1})) \right]\]


\[
\leq \max \left\{ \phi_1 \left[ \frac{1}{2} D(x_{2m}, x_{2n}, g(x_{2m}, x_{2n})) + D(x_{2n-1}, x_{2m+1}, g(x_{2n-1}, x_{2m+1})) \right] \right\},
\]

\[
\phi_2 \left[ D(x_{2m}, x_{2m+1}, g(x_{2m}, x_{2m+1})) \right], \phi_3 \left[ D(x_{2n-1}, x_{2n}, g(x_{2n-1}, x_{2n})) \right],
\]

\[
\phi_4 \left[ D(x_{2m}, x_{2n-1}, g(x_{2m}, x_{2n-1})) \right]\}
\]

(3.6)

Thus, we obtain \( \lim_{n \to \infty} D(x_{2n+1}, u, g(x_{2n+1}, u)) = 0 \). Assume that \( u \neq Su \) or \( u \neq Tu \).
Then, by hypothesis, we have

\[ 0 < \inf \{ D(x, u, g(x, u)) + D(x, Sx, g(x, Sx)) + D(y, Ty, g(y, Ty)) : x, y \in X \} \]
\[ \leq \inf \{ D(x_{2n+1}, u, g(x_{2n+1}, u)) + D(x_{2n+1}, Sx_{2n+1}, g(x_{2n+1}, Sx_{2n+1})) \]
\[ + D(x_{2n+2}, Tx_{2n+2}, g(x_{2n+2}, Tx_{2n+2})) : n \in \mathbb{N} \} \]
\[ = \inf \{ D(x_{2n+1}, u, g(x_{2n+1}, u)) + D(x_{2n+1}, x_{2n+2}, g(x_{2n+1}, x_{2n+2})) \]
\[ + D(x_{2n+2}, x_{2n+3}, g(x_{2n+2}, x_{2n+3})) : n \in \mathbb{N} \} \]
\[ = 0. \]

This is a contradiction. Therefore, we have \( u = Su = Tu \).

On the other hand, we can prove the existence of a unique common fixed point of \( S \) and \( T \) by a method similar to that of Theorem 3.1. \( \square \)

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References


Hee Soo Park: Department of Applied Mathematics, Changwon National University, Changwon 641-773, Korea
*E-mail address: pheesoo@changwon.ac.kr*

Jeong Sheok Ume: Department of Applied Mathematics, Changwon National University, Changwon 641-773, Korea
*E-mail address: jsume@changwon.ac.kr*