We will define the extension of \( q \)-Hurwitz zeta function due to Kim and Rim (2000) and study its properties. Finally, we lead to a useful new integral representation for the \( q \)-zeta function.

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1. Introduction. Let \( 0 < q < 1 \) and for any positive integer \( k \), define its \( q \)-analogue
\[
[k]_q = \frac{(1 - q^k)}{(1 - q)}.
\]

Let \( C \) be the field of complex numbers. The \( q \)-zeta function due to T. Kim was defined as
\[
\zeta_q^{(h)}(s) = \sum_{n=1}^{\infty} \left( \frac{q^n}{n!} \right) t^n + (q - 1) \frac{1 - s + h}{1 - s} \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^{s-1}}
\]
for any \( s, h \in C \) (cf. [3, 4]). This function can be considered on the spectral zeta function of the quantum group \( SU_q(2) \) (cf. [2, 4]). Also, the \( q \)-zeta function \( \zeta_q^{(h)}(s) \) was studied at negative integers (see [4]). In this note, we lead to a useful new integral representation for the \( q \)-zeta function \( \zeta_q^{(h)}(s) \). Finally, we define the extension of \( q \)-Hurwitz zeta function, and study its properties.

2. \( q \)-zeta functions. For \( q \in C \) with \( |q| < 1 \), we define \( q \)-Bernoulli polynomials as follows:
\[
F_q^{(h)}(t, x) = \sum_{n=0}^{\infty} \frac{\beta_{n,q}^{(h)}(x)}{n!} t^n
\]
\[
= e^{(1/(1-q))t} \sum_{j=0}^{\infty} \frac{j + h}{[j + h]_q} (-1)^j q jx \left( \frac{1}{1-q} \right) ^{j} \frac{t^j}{j!}
\]
\[
= -t \sum_{t=0}^{\infty} q^{h+1} e^{(1+q)q^t} + (1 - q) h \sum_{t=0}^{\infty} q^{h} e^{(1+q)q^t}
\]
for \( h \in Z, x \in C \) (cf. [2, 4]). In the case \( x = 0 \), \( \beta_{n,q}^{(h)}(0) \) will be called the \( q \)-Bernoulli numbers (cf. [4]). By (2.1), we easily see that
\[ \beta^{(h)}_{n,q}(x) = \sum_{j=0}^{m} \binom{m}{j} [x]_{q}^{n-j} q^{j} \beta^{(h)}_{j,q} \]
\[ = \left( \frac{1}{1-q} \right)^{n} \sum_{j=0}^{n} \binom{n}{j} (-1)^{j} \frac{j+h}{j+1} q^{j} \] (cf. [2]),

where \( \binom{n}{j} \) is a binomial coefficient.

Thus we note that
\[ q^{h} (q^{\beta^{(h)}} + 1)^{n} - \beta^{(h)}_{n,q} = \delta_{1,n}, \]

where we use the usual convention about replacing \( (\beta^{(h)})^{n} \) by \( \beta^{(h)}_{n,q} \) and \( \delta_{1,n} \) is the Kronecker symbol.

**Example 2.1.**
\[ \beta^{(2)}_{0} = 2 \left[ \frac{2}{2} \right], \quad \beta^{(2)}_{1} = -\frac{2q+1}{2}[3][3], \quad \beta^{(2)}_{2} = \frac{2q^{2}}{[3][4]}, \quad \beta^{(2)}_{3} = -\frac{q^{2}(q-1)(2[3]_{q}+q)}{[3][4][5]}, \ldots \]

Let \( F^{(h)}_{q}(t) = \sum_{n=0}^{\infty} \beta^{(h)}_{n,q}/n! t^{n} \). Then we easily see that
\[ F^{(h)}_{q}(x,t) = e^{[x]_{q}t} F^{(h)}_{q}(q^{x}t) \]
\[ = -t \sum_{l=0}^{\infty} q^{l(h+1)+x} e^{[l+x]_{q}t} + (1-q) h \sum_{l=0}^{\infty} q^{l} e^{[l+x]_{q}t}. \]

By (2.1) and (2.5), we note that
\[ e^{-t} F^{(h)}_{q}(-qt) = qt \sum_{l=0}^{\infty} q^{l(h+1)} e^{-(l+1)_{q}t} + (1-q) h \sum_{l=0}^{\infty} q^{l} e^{-(l+1)_{q}t}. \]

Thus we have
\[ \frac{1}{\Gamma(s)} \int_{0}^{\infty} q^{ht-2} e^{-t} F^{(h)}_{q}(-qt) dt = \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_{q}^{s}} + (q-1) \frac{h+1-s}{1-s} \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_{q}^{s-1}}. \]

For \( h,s \in \mathbb{C} \), we define the \( q \)-zeta function as follows:
\[ \zeta_{q}^{(h)}(s) = \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_{q}^{s}} + (q-1) \frac{h+1-s}{1-s} \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_{q}^{s-1}} \] (cf. [1, 4]).

Note that \( \zeta_{q}^{(h)}(s) \) is a meromorphic function for \( \text{Re}(s) > 1 \).

Let \( \Gamma(s) \) be the gamma function and let \( \mathbb{Z} \) be the set of integers. By (2.3), (2.7), and (2.8), we obtain the following.

For \( h,n(>1) \in \mathbb{Z} \), we have
\[ \zeta_{q}^{(h)}(1-n) = -\frac{q^{h}(q^{\beta^{(h)}} + 1)^{n}}{n} = -\frac{\beta^{(h)}_{n,q}}{n}. \]
Let $x$ be any nonzero positive real number. Then we define the $q$-analogue of Hurwitz zeta function as follows:

$$\zeta_q^{(h)}(s,x) = \sum_{n=0}^{\infty} \frac{q^{nh}}{[n+x]_q} + \frac{h+1-s}{1-s} \sum_{n=0}^{\infty} \frac{q^{nh}}{[n+x]_q^{s-1}}$$

for $s, h \in \mathbb{C}$. By (2.5) and (2.10), we easily see that

$$\zeta_q^{(h)}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q^{(h)}(x,-t) \, dt. \quad (2.11)$$

Thus we obtain the following: for $n \in \mathbb{N}$, $h \in \mathbb{Z}$, we have

$$\zeta_q^{(h)}(1-n) = -\frac{\beta_q^{(h)}(x)}{n}$$

because

$$\zeta_q^{(h)}(s,x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \beta_q^{(h)}(x) \frac{1}{\Gamma(s)} \int_0^\infty t^{s+n-2} \, dt. \quad (2.13)$$

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**References**


T. Kim: Institute of Science Education, Kongju National University, Kongju 314-701, Korea

*E-mail address:* tkim@kongju.ac.kr

L. C. Jang: Department of Mathematics and Computer Science, Konkuk University, Choongju 380-701, Korea

*E-mail address:* leechae.jang@kku.ac.kr

S. H. Rim: Department of Mathematics Education, Kyungpook National University, Daegu 702-701, Korea

*E-mail address:* shrim@knu.ac.kr