MULTIVALENT FUNCTIONS AND $Q_K$ SPACES

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We give a criterion for $q$-valent analytic functions in the unit disk to belong to $Q_K$, a Möbius-invariant space of functions analytic in the unit disk in the plane for a nondecreasing function $K : [0, \infty) \to [0, \infty)$, and we show by an example that our condition is sharp. As corollaries, classical results on univalent functions, the Bloch space, BMOA, and $Q_p$ spaces are obtained.

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1. Introduction. For analytic univalent function $f$ in the unit disk $\Delta$, Pommerenke [8] proved that $f \in H^1_{\log}$ if and only if $f \in \text{BMOA}$, which easily implies a result of Baernstein II [4] about univalent Bloch functions: if $g(z) \neq 0$ is an analytic univalent function in $\Delta$, then $\log g \in \text{BMOA}$. We know that Pommerenke’s result mentioned above was generalized to $Q_p$ spaces for all $p, 0 < p < \infty$, by Aulaskari et al. (cf. [2, Theorem 6.1]). Their result can be stated as follows.

**Theorem 1.1.** Let $f$ be an analytic function in $\Delta$ such that

$$\int \left| \frac{w-w_0}{1-\overline{w}_0 w} \right| n(w,f) \, dA(w) \leq A < \infty,$$

for all $w_0 \in \mathbb{C}$, where $n(w,f)$ denotes the number of roots of the equation $f(z) = w$ in $\Delta$ counted according to their multiplicity and $dA(z)$ is the Euclidean area element on $\Delta$. Then $f \in \mathcal{B} (\mathcal{B}_0)$ if and only if $f \in Q_p (Q_{p,0})$ for all $p \in (0, \infty)$.

Here, $Q_p$ and its subspace $Q_{p,0}, 0 < p < \infty$, denote the spaces of analytic functions $f$ in $\Delta$ defined, respectively, as follows (cf. [1, 3]):

$$Q_p = \left\{ f : f \text{ analytic in } \Delta, \sup_{a \in \Delta} \int_{\Delta} |f'(z)|^2 (g(z,a))^p \, dA(z) < \infty \right\},$$

$$Q_{p,0} = \left\{ f \in Q_p : \lim_{|a| \to 1} \int_{\Delta} |f'(z)|^2 (g(z,a))^p \, dA(z) = 0 \right\},$$

where $g(z,a) = \log 1 / |\varphi_a(z)|$ is a Green’s function in $\Delta$ with pole at $a \in \Delta$, and $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ is a Möbius transformation of $\Delta$.

We know that $Q_1 = \text{BMOA}$, the space of all analytic functions of bounded mean oscillation (cf. [5]), and for each $p \in (1, \infty)$, the space $Q_p$ is the Bloch space $\mathcal{B}$ (cf. [1]), which...
is defined as follows:

$$\mathcal{B} = \left\{ f : f \text{ analytic in } \Delta, \| f \|_\mathcal{B} = \sup_{z \in \Delta} (1 - |z|^2) |f'(z)| < \infty \right\}. \quad (1.3)$$

Similar to the above we have $Q_{1,0} = \text{VMOA},$ the space of all analytic functions of vanishing mean oscillation (cf. [5]), and $Q_{p,0} = \mathcal{B}_0$ for all $p \in (1, \infty),$ where $\mathcal{B}_0$ denotes the little Bloch space defined by

$$\mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{|z| \to 1} (1 - |z|^2) |f'(z)| = 0 \right\}. \quad (1.4)$$

In the present paper, we consider a more general space $Q_K$ (see below) and show that all the above-mentioned results are true for space $Q_K.$ Our contribution gives an extended version of Pommerenke’s theorem, which is also a slight improvement of all the above results, and the proof presented here is independently developed.

Let $K : [0, \infty) \to [0, \infty)$ be a right-continuous and nondecreasing function. Recall that the space $Q_K$ consists of analytic functions $f$ in $\Delta$ for which

$$\| f \|_{Q_K}^2 = \sup_{a \in \Delta} \int_{\Delta} |f'(z)|^2 K(g(z,a)) dA(z) < \infty; \quad (1.5)$$

$f \in Q_K$ belongs to the space $Q_{K,0}$ if

$$\int_{\Delta} |f'(z)|^2 K(g(z,a)) dA(z) \to 0, \quad |a| \to 1. \quad (1.6)$$

Modulo constants, $Q_K$ is a Banach space under the norm defined in (1.5). It is clear that $Q_K$ is Möbius-invariant and a subspace of the Bloch space $\mathcal{B}$ (cf. [6]). For $0 < p < \infty,$ $K(t) = t^p$ gives the space $Q_p.$ Choosing $K(t) = 1,$ we get the Dirichlet space $\mathfrak{D}.$

By [6, Proposition 2.1] we know that if the integral

$$\int_0^{1/e} K \left( \log \frac{1}{\rho} \right) \rho \, d\rho = \int_1^\infty K(t)e^{-2t} \, dt \quad (1.7)$$

is divergent, then the space $Q_K$ is trivial; that is, the space $Q_K$ contains only constant functions. From now on, we assume that the function $K : [0, \infty) \to [0, \infty)$ is right-continuous and nondecreasing and that the integral (1.7) is convergent. Without loss of generality, we can assume that $K(1) > 0.$ For a general theory for $Q_K$ spaces, see [6, 11].

2. Main results. A function $f$ analytic in the unit disk is said to be $q$-valent if the equation $f(z) = w$ has never more than $q$ solutions. Let

$$p(\rho) = \frac{1}{2\pi} \int_0^{2\pi} n(\rho e^{i\phi}, f) \, d\phi. \quad (2.1)$$

If

$$\int_0^R p(\rho) d(\rho^2) \leq qR^2, \quad R > 0, \quad (2.2)$$
where \( q \) is a positive number, we say that \( f \) is areally mean \( q \)-valent or circumferentially mean \( q \)-valent, respectively (cf. [7, pages 38 and 144]). It is clear that if \( f \) is circumferentially mean \( q \)-valent, then \( f \) is areally mean \( q \)-valent.

Note that if (1.1) holds, \( f \) will be areally mean \( q \)-valent in \( \Delta \) for some \( q > 0 \). We know that if \( f \) is univalent, then \( f \) must be areally and circumferentially mean 1-valent. Thus, it is natural to conjecture that Pommerenke’s result and Theorem 1.1 are also true for the areally and circumferentially mean \( q \)-valent functions.

We know that the space \( Q_K \) can be nontrivial if \( K \) is not too big at infinity (see condition (1.7)). For such functions \( K \), the properties of \( Q_K \) depend essentially on the behavior of \( K \) near the origin. From [6, Theorems 2.3 and 2.5], we know that \( Q_K = \mathcal{B}(Q_{K,0} = \mathcal{B}_0) \) if and only if

\[
\int_0^1 (1-r)^{-2} K \left( \log \frac{1}{1-r} \right) r dr < \infty. \tag{2.4}
\]

A natural idea is to look for an integral condition which is weaker than that given by (2.4) such that \( f \in \mathcal{B}(\mathcal{B}_0) \) if and only if \( f \in Q_K(Q_{K,0}) \) for some special \( f \). For the areally mean \( q \)-valent case, we present the main result in this paper as follows.

**Theorem 2.1.** Let \( f \) be an areally mean \( q \)-valent function in \( \Delta \). If

\[
\int_0^1 \left( \log \frac{1}{1-r} \right)^2 (1-r)^{-1} K \left( \log \frac{1}{1-r} \right) r dr < \infty, \tag{2.5}
\]

then

(i) \( f \in \mathcal{B} \) if and only if \( f \in Q_K \);

(ii) \( f \in \mathcal{B}_0 \) if and only if \( f \in Q_{K,0} \).

Note that (2.4) implies (2.5) since \( (\log 1/(1-r))^2 \leq 4 e^{-2} / (1-r) \) for \( 0 < r < 1 \), but the converse is not true. For example, \( K(t) = t \) gives that (2.5) holds but (2.4) fails. By [6, Theorems 2.3 and 2.5], (2.5) is also necessary for Theorem 2.1(i) and (ii) in case \( f \) is an areally mean \( q \)-valent function in \( \Delta \).

In the light of the following example it is impossible to drop the assumption of areally mean \( q \)-valence of the functions \( f \) in Theorem 2.1. Indeed, choose \( K_1(t) = t^{2\alpha-1} \) and

\[
f_1(z) = \sum_{j=1}^{\infty} 2^{-j(1-\alpha)} z^{2^j}, \quad \frac{1}{2} < \alpha < 1. \tag{2.6}
\]

It is easy to see that \( f_1 \in \mathcal{B} \) and (2.5) holds for \( K_1 \). Since \( f_1 \) has a gap series representation, \( f_1 \) is not an areally mean \( q \)-valent in \( \Delta \). The following argument shows that \( f \notin Q_{K_1} \).
For \( r \in [3/4, 1) \), we find \( k \) so that \( 1/2 \leq 2^k (1 - r) < 1 \). Using the inequality \( \log r \geq 2(r - 1) \), \( 1/2 < r < 1 \), we see that

\[
\int_0^{2\pi} |f'_1(re^{i\theta})|^2 d\theta = 2\pi \sum_{j=1}^{\infty} 2^{2j+1} r^{2j+1-2} \geq 2\pi (1 - r)^{-2\alpha} \sum_{j=1}^{\infty} (2^j (1 - r))^{2\alpha} \exp(-2^{j+2}(1 - r)) \]

\[
= C(\alpha) (1 - r)^{-2\alpha}. \tag{2.7}
\]

Hence

\[
\sup_{a \in \Delta} \int \int_{\Delta} |f'_1(z)|^2 K_1(g(z,a)) dA(z) \geq \int \int_{\Delta} |f'_1(z)|^2 K_1\left(\log \frac{1}{|z|}\right) dA(z) \geq \int_0^1 K\left(\log \frac{1}{r}\right) r \, dr \int_0^{2\pi} |f'_1(re^{i\theta})|^2 d\theta \geq C(\alpha) \int_{3/4}^1 (1 - r)^{-2\alpha} \left(\log \frac{1}{r}\right)^{2\alpha-1} r \, dr. \tag{2.8}
\]

Since the last integral is divergent, we conclude that \( f_1 \notin Q_K \).

**Theorem 2.2.** Let \( f \) be a circumferentially mean \( q \)-valent and nonvanishing function in \( \Delta \). If (2.5) holds, then \( \log f \in Q_K \).

It is clear that the integral in (2.5) is convergent for \( K(t) = t^p, \, p > 0 \). Thus, we have the following result which extends Theorem 1.1.

**Corollary 2.3.** Let \( f \) be an areally mean \( q \)-valent function in \( \Delta \), \( 0 < p < \infty \). Then

(i) \( f \in \mathcal{B} \) if and only if \( f \in Q_p \);

(ii) \( f \in \mathcal{B}_0 \) if and only if \( f \in Q_{p,0} \).

3. **Proofs.** In the proofs of Theorems 2.1 and 2.2, we need two lemmas, the first one can be considered as a generalization of a result of Pommerenke (cf. [9, page 174]).

**Lemma 3.1.** Let \( f \) be areally mean \( q \)-valent in \( \Delta \). Then

\[
\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \leq \frac{4q\pi (M(\sqrt{r},f))^2}{1-r}, \quad \frac{1}{2} < r < 1, \tag{3.1}
\]

where \( M(r,f) = \sup_{|z|=r} |f(z)|, \, 0 < r < 1 \).
\textbf{Proof.} If \(1/2 < r < 1\), we obtain
\[
\iint_{|z|<\sqrt{r}} |f'(z)|^2 dA(z) = \int_0^r \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \, dr
\geq \frac{1}{4} (1-r) \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta.
\] (3.2)

Since \(f\) is areally mean \(q\)-valent, we deduce that
\[
\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \leq \frac{4}{1-r} \int_{|z|<\sqrt{r}} |f'(z)|^2 dA(z)
\leq \frac{4}{1-r} \int_{|w|<M(\sqrt{r},f)} n(w,f) \, dA(w)
\leq \frac{4q\pi(M(\sqrt{r},f))^2}{1-r},
\] (3.3)

which proves Lemma 3.1.

\textbf{Lemma 3.2.} Let \(K\) be defined as in Section 1. Then
\begin{enumerate}
\item[(i)] \(Q_{K,0} \subset B_0\);
\item[(ii)] an analytic function \(f\) belongs to \(B_0\) if and only if there exists an \(r \in (0,1)\) such that
\[
\lim_{|a| \to 1} \iint_{\Delta(a,r)} |f'(z)|^2 K(g(z,a)) \, dA(z) = 0,
\] (3.4)
\end{enumerate}

where \(\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}\).

\textbf{Proof.} See [6, Thereom 2.4].

Now we turn to give the proofs of our main theorems.

\textbf{Proof of Theorem 2.1.} We first prove (i). Since \(Q_K \subset B\), it suffices to prove that if a Bloch function \(f\) is areally mean \(q\)-valent in \(\Delta\), then \(f \in Q_K\). We use the change of variable \(w = \varphi_a(z)\) to deduce that
\[
\iint_{\Delta(a,1/2)} |f'(z)|^2 K(g(z,a)) \, dA(z)
= \iint_{\Delta(a,1/2)} |(f(z) - f(a))'|^2 K \left( \log \frac{1}{|\varphi_a(z)|} \right) \, dA(z)
= \iint_{1/2<|w|<1} |(f \circ \varphi_a(w) - f(a))'|^2 K \left( \log \frac{1}{|w|} \right) \, dA(w)
= \int_{1/2}^1 K \left( \log \frac{1}{r} \right) r \int_0^{2\pi} |(f \circ \varphi_a(re^{i\theta}) - f(a))'|^2 d\theta \, dr.
\] (3.5)
It is known that if $g \in \mathcal{B}$, then
\[
|g(z) - g(0)| \leq \frac{1}{2} \|g\|_{\mathcal{B}} \log \frac{1 + |z|}{1 - |z|}.
\] (3.6)

Choosing $g = f \circ \varphi_a - f(a)$ and observing that $\|g\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$, we obtain
\[
M(r, f \circ \varphi_a - f(a)) \leq \frac{1}{2} \|f\|_{\mathcal{B}} \log \frac{1 + r}{1 - r}.
\] (3.7)

It follows from (3.5) and Lemma 3.1 that
\[
\iint_{\Delta \setminus \Delta(a,1/2)} |f'(z)|^2 K(g(z,a))dA(z)
= \int_{1/2}^{1} K\left(\log \frac{1}{r}\right) r \int_{0}^{2\pi} |(f \circ \varphi_a(re^{i\theta}) - f(a))'|^2 d\theta dr
\leq 4q\pi \int_{1/2}^{1} K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1 - r)^{-1} r dr
\leq q\pi C \|f\|_{\mathcal{B}}^2 \int_{1/2}^{1} K\left(\log \frac{1}{r}\right) \left(\log \frac{1}{1 - r}\right)^2 (1 - r)^{-1} r dr.
\] (3.8)

On the other hand, we have
\[
\iint_{\Delta(a,1/2)} |f'(z)|^2 K(g(z,a))dA(z)
\leq \|f\|_{\mathcal{B}}^2 \iint_{\Delta(0,1/2)} (1 - |z|^2)^{-2} K(g(z,a))dA(z)
= \|f\|_{\mathcal{B}}^2 \int_{0}^{1/2} K\left(\log \frac{1}{|w|}\right) dA(w)
\leq 4\pi \|f\|_{\mathcal{B}}^2 \int_{0}^{1/2} K\left(\log \frac{1}{r}\right) r dr.
\] (3.9)

Combining the upper bounds given by (3.8), (3.9), and (2.5), we see that $f \in Q_K$, which proves part (i) of Theorem 2.1.

To prove (ii), we assume that $f$ is an areally mean $q$-valent function in $\Delta$ which is also in $\mathcal{B}_0$. By Lemma 3.2(i), it suffices to prove that $f \in Q_{K,0}$. By Lemma 3.2(ii), there exists an $r_0$, $1/2 < r_0 < 1$, such that
\[
\lim_{|a| \to 1} \iint_{\Delta(a,r_0)} |f'(z)|^2 K(g(z,a))dA(z) = 0.
\] (3.10)

Now we show that
\[
\lim_{|a| \to 1} \iint_{\Delta(a,r_0)} |f'(z)|^2 K(g(z,a))dA(z) = 0.
\] (3.11)
By the proof of part (i) and assumption (2.5), we see that

\[ \int_{\Delta \setminus \Delta(a,r_0)} \left| f'(z) \right|^2 K(g(z,a)) dA(z) \]
\[ = \int_{r_0}^1 K \left( \log \frac{1}{r} \right) r^{2\pi} \left| (f \circ \varphi_a(r e^{i\theta}) - f(a))' \right|^2 d\theta dr \]
\[ \leq 4q\pi \int_{r_0}^1 K \left( \log \frac{1}{r} \right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr \]
\[ \leq q\pi \| f \|^2_{\mathfrak{B}} \int_{r_0}^1 K \left( \log \frac{1}{r} \right) \left( \log \frac{1+r}{1-r} \right)^2 (1-r)^{-1} r dr < \infty \] (3.12)

for all \( a \in \Delta \). Thus, for any given \( \varepsilon > 0 \), there exists an \( r_1, r_0 < r_1 < 1 \), such that

\[ \int_{r_1}^1 K \left( \log \frac{1}{r} \right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr < \varepsilon \] (3.13)

for all \( a \in \Delta \). Hence, what we need to prove is that

\[ \lim_{|a| \to 1} \int_{r_0}^{r_1} K \left( \log \frac{1}{r} \right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr = 0. \] (3.14)

In fact, we have

\[ \int_{r_0}^{r_1} K \left( \log \frac{1}{r} \right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr \]
\[ \leq C(r_0,r_1) K \left( \log \frac{1}{r_0} \right) (M(r_2, f \circ \varphi_a - f(a)))^2, \] (3.15)

where \( r_2 = \sqrt{r_1} \) and \( C(r_0,r_1) \) is a constant depending on \( r_0 \) and \( r_1 \). Define \( f_t(z) = f(tz) \) for \( 0 < t < 1 \) and then

\[ (M(r_2, f \circ \varphi_a - f(a)))^2 \]
\[ \leq 2 \left( \frac{1}{4} \| f - f_t \|^2_{\mathfrak{B}} \left( \log \frac{1+r_2}{1-r_2} \right)^2 + (M(r_2, f_t \circ \varphi_a - f_t(a)))^2 \right). \] (3.16)

Since \( f \in \mathfrak{B}_0, \| f - f_t \|^2_{\mathfrak{B}} \to 0, t \to 1 \). Also,

\[ \max_{|z| = r_2} | f_t \circ \varphi_a(z) - f_t(a) | \leq \frac{1-|a|^2}{(1-r_2)^2} \max_{|w| = t} | f'(w) |, \] (3.17)

which implies that

\[ \lim_{|a| \to 1} M(r_2, f_t \circ \varphi_a - f_t(a)) = 0. \] (3.18)
Thus we have (3.14). Hence

$$\lim_{|a| \to 1} \int_{\Delta} |f'(z)|^2 K(g(z,a)) dA(z) = 0,$$

which shows that $f \in Q_{K,0}$. The proof of Theorem 2.1 is complete.

**Proof of Theorem 2.2.** Assume that $f$ is a nonvanishing circumferentially mean $q$-valent function in $\Delta$. According to [7, Theorem 5.1], we have $\log f \in \mathcal{B}$. From [7, Lemma 5.2] and the argument in the beginning of the proof of [7, Theorem 5.1], we see that we can define a single-valued branch of $f(z)^{1/q}$ which is circumferentially mean 1-valent in $\Delta$ and such that on each circle $\{|w| = R\}$ there exists a point which is not assumed by $f(z)^{1/q}$. It follows that

$$\int_{-\infty}^{\infty} n\left(\log \rho + i\phi, \frac{1}{q} \log f\right) d\phi = \int_{0}^{2\pi} n(\rho e^{i\phi}, f^{1/q}) d\phi \leq 2\pi,$$

$$\int_{|w| < R} n(w, \log f) dA(w) \leq 4\pi Rq,$$

which means that $\log f$ is areally mean $q_1$-valued in $\Delta$ for some $q_1 > 0$. It follows from Theorem 2.1 that $\log f \in Q_K$.

4. Further discussion. In [10] we studied the conditions for analytic univalent Bloch function $f$ to belong to $Q_k$ spaces. The log-order of the function $K(r)$ is defined as

$$\rho = \lim_{r \to \infty} \frac{\log^+ \log^+ K(r)}{\log r},$$

where $\log^+ x = \max\{\log x, 0\}$, and if $0 < \rho < \infty$, the log-type of the function $K(r)$ is defined as

$$\sigma = \lim_{r \to \infty} \frac{\log^+ K(r)}{r^p}.\tag{4.2}$$

**Theorem 4.1.** Let $f$ be an analytic univalent function in $\Delta$ and let $K : [0, \infty) \to [0, \infty)$ satisfy that $K(t) = O((t \log 1/t)^p)$ as $t \to 0$ for some $p > 0$. If the log-order $\rho$ and the log-type $\sigma$ of $K$ satisfy one of the conditions

(i) $0 \leq \rho < 1$,
(ii) $\rho = 1$ and $\sigma < 2$,
then $f \in \mathcal{B}$ if and only if $f \in Q_K$.

We note that Theorem 4.1 can be viewed as a consequence of Theorem 2.1. In fact, conditions (i) and (ii) of Theorem 4.1 show that the space $Q_K$ is not trivial. That is, the integral (1.7) is convergent in this case. Suppose that $K(t) = O((t \log 1/t)^p)$, $t \to 0$. There exist an $r_0 \in (1/2, 1)$ and a constant $C > 0$ such that both $\log 1/r \leq 2(1-r)$ and

$$K\left(\log \frac{1}{r}\right) \leq C \left(\log \frac{1}{r} \log \left(\log \frac{1}{r}\right)^{-1}\right)^p \tag{4.3}$$
hold for \( r_0 < r < 1 \). Thus

\[
\int_0^1 \left( \log \frac{1}{1-r} \right)^2 (1-r)^{-1} K \left( \log \frac{1}{r} \right) r \, dr
= \int_{r_0}^{r} + \int_{r_0}^1 \left( \log \frac{1}{1-r} \right)^2 (1-r)^{-1} K \left( \log \frac{1}{r} \right) r \, dr
\leq \left( \log \frac{1}{1-r_0} \right)^2 (1-r_0)^{-1} \int_{r_0}^1 K \left( \log \frac{1}{r} \right) r \, dr
\]
\[
+ C \int_{r_0}^1 \left( \log \frac{1}{1-r} \right)^2 (1-r)^{-1} \left( \log \frac{1}{r} \log \left( \log \frac{1}{r} \right)^{-1} \right) r \, dr
\leq C_1 + C_2 \int_{r_0}^1 \left( \log \frac{1}{1-r} \right)^{2+p} (1-r)^{p-1} r \, dr
\leq C_1 + C_2 \int_{r_0}^\infty e^{-p s} s^{2+p} ds
\leq C_1 + C_2 p^{-3-p} \Gamma(3+p) < \infty.
\]

(4.4)

For a general analytic function \( f \), we have the following theorem.

**THEOREM 4.2.** Suppose that (2.5) holds. If

\[
\sup_{a \in \Delta} \iint_{|z|<r} |(f \circ \varphi_a(z))'|^2 dA(z) = O \left( \left( \log \frac{1}{1-r} \right)^2 \right),
\]

(4.5)

then

(i) \( f \in B \) if and only if \( f \in Q_K \);

(ii) \( f \in B_0 \) if and only if \( f \in Q_{K,0} \).

**PROOF.** We know that

\[
\int_0^{2\pi} \left| (f \circ \varphi_a(re^{i\theta}))' \right|^2 d\theta \leq \frac{4}{1-r} \iint_{|z|<\sqrt{r}} \left| (f \circ \varphi_a(z))' \right|^2 dA(z)
\]
\[
\leq \frac{1}{1-r} O \left( \left( \log \frac{1}{1-\sqrt{r}} \right)^2 \right)
\leq \frac{C}{1-r} \left( \log \frac{1}{1-r} \right)^2.
\]

(4.6)

The proof can be completed by an argument similar to that used in the proof of Theorem 2.1.

\[\square\]

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