ON SOME PROPERTIES OF BANACH OPERATORS. II

A. B. THAHEEM and A. R. KHAN

Received 30 September 2003

Using the notion of a Banach operator, we have obtained a decompositional property of a Hilbert space, and the equality of two invertible bounded linear multiplicative operators on a normed algebra with identity.

2000 Mathematics Subject Classification: 46C05, 47A10, 47A50, 47H10.

1. Introduction. This paper is a continuation of our earlier work [7] on Banach operators. We recall that if $X$ is a normed space and $\alpha : X \to X$ is a mapping, then following [4], $\alpha$ is said to be a Banach operator if there exists a constant $k$ such that $0 \leq k < 1$ and $\|\alpha^2(x) - \alpha(x)\| \leq k\|\alpha(x) - x\|$ for all $x \in X$. Banach operators are generalizations of contraction maps and play an important role in the fixed point theory; their consideration goes back to Cheney and Goldstein [2] in the study of proximity maps on convex sets (see [4] and the references therein).

In [7], we established some decompositional properties of a normed space using Banach operators. We showed that if $\alpha$ is a linear Banach operator on a normed space $X$, then $N(\alpha - 1) = N((\alpha - 1)^2)$, $N(\alpha - 1) \cap R(\alpha - 1) = \{0\}$ and in case $X$ is finite dimensional, we get the decomposition $X = N(\alpha - 1) \oplus R(\alpha - 1)$, where $N(\alpha - 1)$ and $R(\alpha - 1)$ denote the null space and the range space of $(\alpha - 1)$, respectively, and 1 denotes the identity operator on $X$. In [7, Proposition 2.3], we proved a decompositional property of a general bounded linear operator on a Hilbert space, namely, if $\alpha$ is a bounded linear operator on a Hilbert space $H$ such that $\alpha$ and $\alpha^*$ have common fixed points, then $N(\alpha - 1) + R(\alpha - 1)$ is dense in $H$.

In this paper, also we prove some properties of Banach operators on a Hilbert space. We show (Proposition 2.1) that if $\alpha$ is a bounded linear Banach operator on a Hilbert space $H$ such that the sets of fixed points of $\alpha$ and $\alpha^*$ are the same, then $H$ admits a decomposition $H = N(\alpha - 1) \oplus M$, where $M = R(\alpha - 1)$, $(R(\alpha - 1)$ denotes the closure of $R(\alpha - 1))$. It follows as a corollary of Proposition 2.1 that $\alpha$ commutes with both orthogonal projections onto $N(\alpha - 1)$ and onto $M$.

As in [7], we also study the operator equation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ for a pair of invertible bounded linear multiplicative Banach operators $\alpha$ and $\beta$ on a normed algebra $X$ with identity, where $c$ is an appropriate real or complex number. We prove the following result (Proposition 2.3): assume that $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ for all $x \in X$, where $c$ is a real or complex number such that $|c| \geq 1$, $\|\alpha\|^2 \leq |c|/2$, $\|\beta\|^2 \leq |c|/2$. If $\beta$ is inner, then $\alpha = \beta$. We briefly recall that this operator equation has been extensively studied for automorphisms on von Neumann algebras. We refer to [1, 5, 6] for more details about this operator equation.
2. The results

**Proposition 2.1.** Let $\alpha$ be a bounded linear Banach operator on a Hilbert space $H$ such that the sets of fixed points of $\alpha$ and $\alpha^*$ are the same. Then the following hold:

(i) $N(\alpha - 1) \perp R(\alpha - 1)$,
(ii) $H = N(\alpha - 1) \oplus M$, where $M = R(\alpha - 1)$.

**Proof.** To prove (i), let $x \in N(\alpha - 1)$ and $y \in R(\alpha - 1)$. Then $\alpha(x) = x$ and $y = \alpha(z) - z$ for some $z \in H$. Therefore, $\alpha^*(x) = x$ and hence

$$\langle x, y \rangle = \langle x, \alpha(z) - z \rangle = \langle x, \alpha(z) \rangle - \langle x, z \rangle = \langle \alpha^*(x), z \rangle - \langle x, z \rangle = \langle x, z \rangle - \langle x, z \rangle = 0.$$

(2.1)

Thus $N(\alpha - 1) \perp R(\alpha - 1)$.

To prove (ii), it is enough to show that $N(\alpha - 1) = M^\perp$. By (i) and the continuity of $\alpha$, $N(\alpha - 1) \subseteq M^\perp$. Conversely, assume that $z \in M^\perp$. Then $\langle z, y \rangle = 0$ for all $y \in M$; in particular, $\langle z, (\alpha - 1)x \rangle = 0$ for all $x \in H$ because $R(\alpha - 1) \subseteq M$. Thus $\langle z, \alpha(x) \rangle = \langle z, x \rangle$ for all $x \in H$. So, $\langle \alpha^*(z), x \rangle = \langle z, x \rangle$ for all $x \in H$. This shows that $\langle \alpha^*(z) - z, x \rangle = 0$ for all $x \in H$. Therefore, $\alpha^*(z) - z = 0$ or $\alpha^*(z) = z$, that is, $z$ is a fixed point of $\alpha^*$ and hence by assumption, $\alpha(x) = x$, that is, $z \in N(\alpha - 1)$. So, $M^\perp \subseteq N(\alpha - 1)$. Thus $N(\alpha - 1) = M^\perp$ and hence $H = N(\alpha - 1) \oplus M$. 

**Corollary 2.2.** Let $\alpha$ be a bounded linear Banach operator on a Hilbert space $H$ such that the sets of fixed points of $\alpha$ and $\alpha^*$ are the same. Then $\alpha$ commutes with both orthogonal projections, onto $N(\alpha - 1)$ and onto $M$.

**Proof.** Since $R(\alpha - 1)$ is $\alpha$-invariant, so is $M$. Also, $M^\perp = N(\alpha - 1)$ is $\alpha$-invariant. Thus $M$ reduces $\alpha$ and hence $\alpha$ commutes with both orthogonal projections, onto $N(\alpha - 1)$ and onto $M$ [3].

It easily follows that the orthogonal projection $P$ onto $N(\alpha - 1)$ is the largest orthogonal projection such that $\alpha P = P$.

We conclude this paper with a result about an operator equation similar to the one considered in [7].

**Proposition 2.3.** Let $\alpha, \beta$ be invertible bounded linear multiplicative Banach operators on a normed algebra $X$ with identity such that $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ for all $x \in X$, where $c$ is a real or complex number with $|c| \geq 1$, $\|\alpha\|^2 \leq |c|/2$, $\|\beta\|^2 \leq |c|/2$. If $\beta$ is invertible, then $\alpha = \beta$.

**Proof.** It follows from [7, Proposition 3.2] that $\alpha$ and $\beta$ commute. Therefore,

$$\begin{align*}
(\alpha \beta - c)(\beta^{-1} - \alpha^{-1})(x) &= \alpha(x) - \alpha \beta \alpha^{-1}(x) - c \beta^{-1}(x) + c \alpha^{-1}(x) \\
&= \alpha(x) - \beta(\alpha(x) - c \beta^{-1}(x) + c \alpha^{-1}(x)) \\
&= \alpha(x) - \beta(\alpha^{-1}(x)) - c \beta^{-1}(x) + c \alpha^{-1}(x) \\
&= \alpha(x) - \beta(x) - c \beta^{-1}(x) + c \alpha^{-1}(x) \\
&= (\alpha(x) + c \alpha^{-1}(x)) - (\beta(x) + c \beta^{-1}(x)) = 0.
\end{align*}$$

(2.2)
Put \((\beta^{-1} - \alpha^{-1})(x) = y\). Then we obtain \((\alpha \beta - c)(y) = 0\), that is, \(\alpha \beta(y) = cy\). Therefore, by assumption, we get \(|c| \|y\| = \|cy\| = \|\alpha \beta(y)\| \leq \|\alpha\| \|\beta\| \|y\| \leq (|c|/2) \|y\|\), that is, \(|c| \|y\| \leq (|c|/2) \|y\|\). This implies that \(\|y\| = 0\) and hence \((\beta^{-1} - \alpha^{-1})(x) = 0\) for all \(x \in X\), that is, \(\beta^{-1}(x) = \alpha^{-1}(x)\) for all \(x \in X\). Since \(\alpha\) is onto, therefore replacing \(x\) by \(\alpha(x)\), we get \(\beta^{-1}(\alpha(x)) = x\) or \(\alpha(x) = \beta(x)\) for all \(x \in X\). \(\square\)

**Acknowledgments.** The authors are grateful to King Fahd University of Petroleum and Minerals and Saudi Basic Industries Corporation (SABIC) for supporting Fast Track Research Project no. FT-2002/01. We also thank the referees for useful suggestions. The second author is on leave from Bahauddin Zakariya University, Multan, Pakistan, and is indebted to the university’s authorities for granting leave.

**References**


A. B. Thaheem: Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

*E-mail address: athaheem@kfupm.edu.sa*

A. R. Khan: Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

*E-mail address: arahim@kfupm.edu.sa*