We prove that any rigid left Noetherian ring is either a domain or isomorphic to some ring $\mathbb{Z}_{p^n}$ of integers modulo a prime power $p^n$.

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Let $R$ be an associative ring. A map $\sigma : R \to R$ is called a ring endomorphism if $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$ for all elements $a, b \in R$. A ring $R$ is said to be rigid if it has only the trivial ring endomorphisms, that is, identity $\text{id}_R$ and zero $0_R$.

Rigid left Artinian rings were described by Maxson [9] and McLean [11]. Friger [4, 6] has constructed an example of a noncommutative rigid ring $R$ with the additive group $R^+$ of finite Prüfer rank. A characterization for rigid rings of finite rank was obtained by the author in [1]. Some aspects of a ring rigidity has been studied by Suppa [12, 13], Friger [5], and the author [2].

In this paper, we study rigid left Noetherian rings and prove the following theorem.

**Theorem 1.** Let $R$ be a left Noetherian ring. Then $R$ is a rigid ring if and only if $R \cong \mathbb{Z}_{p^n}$ ($p$ is a prime, $t \in \mathbb{N}$) or it is a rigid domain.

All rings are assumed to be associative and, as a rule, with an identity element. For a ring $R$, $N(R)$ will always denote the set of all nil elements of $R$, $\text{char}(R)$ the characteristic, and $\text{Ann}(I) = \{a \in R \mid aI = Ia = \{0\}\}$ the annihilator of $I$ in $R$. If $R$ is a left order in $Q$ (or equivalently, $Q$ is the left quotient ring of $R$), then we will write $Q = Q(R)$. Any unexplained terminology is standard as in [10].

We recall that a ring $R$ is reduced if $r^2 = 0$ implies $r = 0$ for any $r \in R$. Clearly, if $R$ is a rigid reduced ring with an identity element, then either $\text{char}(R) = 0$ or $\text{char}(R) = p$ for some prime $p$.

**Lemma 2.** Let $R$ be a reduced left Goldie ring. If $R$ is rigid, then it is a domain.

**Proof.** Let $R$ be a reduced rigid left Goldie ring. Assume that $R$ is not a domain. From $bx = 0$ (resp., $xb = 0$), where $b, x \in R$, it holds that $(xb)^2 = 0$ (resp., $(bx)^2 = 0$) and thus a right (resp., left) annihilator of every element $b$ in $R$ coincides with $\text{Ann}(b)$. Moreover, in view of [10, Lemma 2.3.2(i)], $\text{Ann}(a)$ is a maximal left annihilator for some $a \in R$.

Assume that the quotient ring $R/\text{Ann}(a)$ contains elements $\overline{x} = x + \text{Ann}(a) \neq \overline{0}$, $\overline{y} = y + \text{Ann}(a)$ such that

$$\overline{x} \overline{y} = \overline{0}$$ (1)


for some \( x, y \in R \). Since \( y \in \text{Ann}(ax) \) and \( \text{Ann}(a) = \text{Ann}(ax) \), we obtain that \( y = 0 \). This means that \( R/\text{Ann}(a) \) is a domain.

By [10, Lemma 2.3.3], \( I_a = Ra \oplus \text{Ann}(a) \) is an essential left ideal of \( R \) and so by [10, Corollary 3.1.8], \( Q(I_a) = Q(R) \). Then the map \( \sigma : I_a \to I_a \) given by \( \sigma(ra) = ra \) \( (r \in R) \) and \( \sigma(\text{Ann}(a)) = \{0\} \) is a nontrivial ring endomorphism of \( I_a \). If \( \sigma : Q(R) \to Q(R) \) is an extension of \( \sigma \) to \( Q(R) \), then

\[
\sigma(r)a = \sigma(ra) = ra
\]

for any \( r \in R \), in which case,

\[
a(\sigma(r) - r) = 0 = (\sigma(r) - r)a.
\]

Since \( \sigma(r) - r = q^{-1}t \) for some regular element \( q \in R \) and some \( t \in R \), we see that

\[
q(\sigma(r) - r) \in \text{Ann}(a).
\]

But \( q \notin \text{Ann}(a) \) and so \( \sigma(r) - r \in \text{Ann}(a) \). This means that \( \sigma(R) \subseteq R \) and \( R \) has a nontrivial ring endomorphism, a contradiction. The lemma is proved.

In the commutative case, we obtain that a commutative reduced rigid Noetherian ring \( R \) of finite exponent is isomorphic to some \( \mathbb{Z}_p \).

Indeed, as it is noted above, \( \text{char}(R) = p \) for some prime \( p \). A map \( \omega : R \to R \) given by the rule \( \omega(x) = x^p \) \( (x \in R) \) is a ring endomorphism of \( R \) and so \( x^p = x \) for all elements \( x \) of \( R \). Assume that \( R \) is not a domain and then it follows that every prime ideal is maximal in \( R \). Hence \( R \) is an Artinian ring by Krull-Akizuki theorem [14, Chapter IV, Section 2, Theorem 2] and by the theorem of [11], \( R \cong \mathbb{Z}_p \), contrary to our assumption. This means that \( R \) is a domain and [9, Theorem 2.5] allows us to state that \( R \cong \mathbb{Z}_p \).

**Remark 3.** Maxson [9] has proved that a rigid commutative domain of prime characteristic \( p \) is isomorphic to \( \mathbb{Z}_p \). Rigid rings of finite rank were studied in [1]. A characterization of rigid commutative domains (in particular, rigid fields) \( R \) of characteristic 0 with the additive group \( R^+ \) of infinite (Prüfer) rank is not known. As it is noted in [8], from the result of Gaifman [7], it holds that there exist rigid Peano fields of arbitrary infinite cardinality. Moreover, it was proved by Dugas and Göbel [3] that each field can be embedded into a rigid field of arbitrary large cardinality.

**Remark 4.** There exist noncommutative rigid Noetherian domains of characteristic 0 (see [4, 6]).

Recall that a map \( d : R \to R \) is called a derivation of \( R \) if

\[
d(x + y) = d(x) + d(y), \quad d(xy) = d(x)y + xd(y)
\]
for all elements \( x, y \in R \). A ring having no nonzero derivations is called differentially trivial (see [1]). Obviously, any differentially trivial ring is commutative.

**Lemma 5.** Let \( R \) be a left Noetherian ring such that \( N(R) \neq \{0\} \). If \( R \) is a rigid ring, then it is isomorphic to some \( \mathbb{Z}_{pt} \).

**Proof.** Suppose that \( R \) is a rigid ring such that \( N = N(R) \neq \{0\} \). Then \( N \subseteq Z(R) \) (see [9, page 96]). Let \( d \) be any nonzero derivation of \( R \). If \( zd(R) = \{0\} \) for all elements \( z \in N \) of the nilpotency indices \( i < n - 1 \) and \( ad(R) \neq \{0\} \) for some element \( a \in N \) of the nilpotency index \( n \), then the rule

\[
\sigma(r) = r + ad(r), \quad r \in R,
\]

(6)
determines a nontrivial ring endomorphism \( \sigma \) of \( R \), a contradiction. Hence

\[
N(R)d(R) = \{0\}
\]

(7)
for every derivation \( d \) of \( R \).

Let \( K_0 = \{a \in N \mid (N \cap Ann(N^2))a = \{0\}\} \). Then \( N \cap Ann(K_0) = N \cap Ann(N^2) \). Assume that \( \delta : R/K_0 \to R/K_0 \) is a nonzero derivation of \( R/K_0 \) and therefore for every \( r \in R \), there is an element \( r_1 \in R \) such that

\[
\delta(r + K_0) = r_1 + K_0.
\]

(8)
Moreover, \( a_1 \notin K_0 \) for some \( a \in R \). Writing \( I \) for the two-sided ideal of \( R \) generated by \( a_1 \), we see that \( (N \cap Ann(N^2))(K_0 + I) \neq \{0\} \). Thus there exists an element \( m_0 \in N \cap Ann(N^2) \) such that \( m_0a_1 \neq 0 \) and so the rule \( g(r) = m_0r_1 \), with \( r \in R \) and \( r_1 \) as in (8), determines a nonzero derivation \( g \) of \( R \). In view of (7) \( g(r)g(t) = 0 \), for any elements \( r, t \in R \) and a map \( \alpha : R \to R \) given by the rule \( \alpha(r) = r + g(r) \), \( (r \in R) \) is a nontrivial ring endomorphism of \( R \), a contradiction with hypothesis. This gives that \( R/K_0 \) is differentially trivial and consequently commutative. Since \( K_0 \subseteq N \) and \( N \subseteq Z(R) \), \( R \) is a Noetherian ring and, as a consequence of [10, Theorem 4.1.9] and [9, Theorem 2.2], \( R \) is an Artinian ring. Finally, by the theorem from [11], \( R \cong \mathbb{Z}_{pt} \) for some prime \( p \) and integer \( t \). This completes the proof. \( \Box \)

**Proof of Theorem 1.** It follows immediately from Lemmas 2 and 5. \( \Box \)

**Corollary 6.** Any rigid simple left Goldie ring \( R \) is a field (or equivalently, any noncommutative simple left Goldie ring has a nontrivial automorphism).  

**Proof.** Since \( N(R) \subseteq Z(R) \), \( R \) is a semiprime ring and so according to [10, Proposition 5.1.5] and Lemma 2, it is a domain. If \( q \) is any element of \( Q(R) \setminus R \) and \( A = q^{-1}Rq \), then \( A \) is a left order in \( Q(R) \). Moreover, \( qAq^{-1} = R \) and so \( A \) and \( R \) are equivalent left orders in \( Q(R) \). By [10, Proposition 5.1.2], \( R \) is a maximal left order in \( Q(R) \) and thus \( A \subseteq R \), which implies \( R \subseteq Z(Q(R)) \), as required. \( \Box \)
REFERENCES


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