SOME REFINEMENTS AND GENERALIZATIONS OF CARLEMAN’S INEQUALITY

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We give some refinements and generalizations of Carleman’s inequality with weaker condition for weight coefficient.

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1. Introduction. The following Carleman’s inequality (see [6, Theorem 334]) is well known, unless \((a_n)\) is null:

\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \tag{1.1}
\]

The constant is the best possible.

There is a vast literature which deals with alternative proofs, various generalizations and extensions, and numerous variants and applications in analysis of inequality (1.1); see [1, 2, 3, 5, 7, 9, 8, 10, 13, 14, 15, 16, 17, 18, 19] and the references given therein. According to Hardy (see [6, Theorem 349]), Carleman’s inequality was generalized as follows. If \(a_n \geq 0, \lambda_n \geq 0, \Lambda_n = \sum_{m=1}^{n} \lambda_m \ (n \in \mathbb{N})\), and \(0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty\), then

\[
\sum_{n=1}^{\infty} \lambda_n \left(\lambda_1^{1/n} \lambda_2^{1/n} \cdots \lambda_n^{1/n}\right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.2}
\]

In [19], Yuan obtained the refined Carleman’s inequality as follows. If \(a_n \geq 0, n = 1, 2, \ldots\), and \(0 < \sum_{n=1}^{\infty} a_n < \infty\), then

\[
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \frac{(1-\beta/n)}{(1+1/n)^{\alpha}} a_n, \tag{1.3}
\]

where \(\alpha, \beta\) satisfy \(0 \leq \alpha \leq 1/\ln 2 - 1, 0 \leq \beta \leq 1 - 2/e, \) and \(e \beta + 2^{1+\alpha} = e\).

Recently, Kim [10] established the following new extension of the refined Hardy’s inequality in the spirit of the property of the power mean of \(n\) distinct positive numbers.
\textbf{Theorem 1.1.} If \(0 < \lambda_{n+1} \leq \lambda_n, a_n \geq 0, \Lambda_n = \sum_{m=1}^{n} \lambda_m, \Lambda_n \geq 1, 0 < p \leq 1, \) and \(0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty, \) then

\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left( \frac{\Lambda_n^\alpha (\Lambda_n - \lambda_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n) \alpha} \right)^p \lambda_n a_n \Lambda_n^{p-1} \left( \sum_{k=1}^{n} \lambda_k c_k^{p} a_k^{p}\right)^{(1-p)/p},
\]

where \(c_k^{\lambda_k} = (\Lambda_{k+1}^{\lambda_k})^{\lambda_k}/(\Lambda_k^{\lambda_k})^{\lambda_{k-1}}, \) and \(\alpha, \beta\) satisfy \(0 \leq \alpha \leq 1/\ln 2 - 1, 0 \leq \beta \leq 1 - 2/e, \) and \(e\beta + 2^{1+\alpha} = e.\)

When \(p = 1\) in inequality (1.4), there exists the following class of new refined Hardy’s inequality:

\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left( \frac{\Lambda_n^\alpha (\Lambda_n - \lambda_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n) \alpha} \right) \lambda_n a_n.
\]

In this paper, we will establish some new refinements and generalizations of Carleman’s inequality. Also, our results correspond to Theorem 1.1 with a weaker condition for weight coefficient.

\section{Results.} The following results are new generalizations of Carleman’s inequality.

\textbf{Theorem 2.1.} Let \(\lambda_n > 0, \nu_n > 0, a_n \geq 0, \Lambda_n = \sum_{m=1}^{n} \lambda_m \nu_m \) \((n \in \mathbb{N}), 0 < p \leq 1, \) and \(0 < \sum_{n=1}^{\infty} \lambda_n \nu_n a_n < \infty.\) If

\[
\frac{\Lambda_{n+1} \Lambda_n}{\Lambda_n + \lambda_n \nu_n} \leq \left( \Lambda_{n-1}^{1-1/p} (\Lambda_n - \lambda_n \nu_n) \right)^{1/p} \left( \Lambda_n - \lambda_n \nu_n \right)^{1/p} \nu_n^{-(\lambda_n - \lambda_n \nu_n) / \nu_n} \nu_{n+1},
\]

then

\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\nu_1} a_2^{\nu_2} \cdots a_n^{\nu_n} \right)^{1/\Lambda_n} \leq \frac{e^p}{p} \sum_{n=1}^{\infty} \left( \frac{\Lambda_n^\alpha (\Lambda_n - \lambda_n \nu_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n \nu_n) \alpha} \right)^p \lambda_n \nu_n a_n^{\nu_n} \Lambda_n^{p-1} \left( \sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^{p}\right)^{(1-p)/p},
\]

where

\[
c_k = \left[ \left( \frac{(\Lambda_k^{1-1/p} \Lambda_{k+1})^{\lambda_k} \nu_k^{\lambda_{k-1}}}{\Lambda_k^{1-1/p} \nu_k^{\lambda_{k-1}}} \right)^{1/\lambda_k \nu_k} \right],
\]

and \(\alpha, \beta\) satisfy \(0 \leq \alpha \leq 1/\ln 2 - 1, 0 \leq \beta \leq 1 - 2/e, \) and \(e\beta + 2^{1+\alpha} = e.\)
Proof. By the power mean inequality [11, page 15], we have
\[
\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n} \leq \left( \frac{1}{n} \sum_{m=1}^{n} q_m \alpha_m^p \right)^{1/p}
\]  
(2.4)
for \( \alpha_m \geq 0, p > 0, \) and \( q_m > 0 (m \in \mathbb{N}) \) with \( \sum_{m=1}^{n} q_m = 1 \). Setting \( c_m > 0, \alpha_m = c_m a_m, \) and \( d_m = \lambda_m v_m / \Lambda_n \), we obtain
\[
(c_1 a_1)^{\lambda_1 v_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 v_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n v_n / \Lambda_n} \leq \left( \sum_{m=1}^{n} \frac{\lambda_m v_m (c_m a_m)^p}{\Lambda_n} \right)^{1/p}.
\]  
(2.5)

Using the above inequality, we have
\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n} \right)^{1/\Lambda_n}
= \sum_{n=1}^{\infty} \lambda_{n+1} \left( \frac{(c_1 a_1)^{\lambda_1 v_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 v_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n v_n / \Lambda_n}}{(c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n})^{1/\Lambda_n}} \right)^{1/p}
\leq \sum_{n=1}^{\infty} \left( \frac{\lambda_{n+1}}{(c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n})^{1/\Lambda_n}} \right)^{1/p} \left( \frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m v_m (c_m a_m)^p \right)^{1/p}.
\]  
(2.6)

By using the inequality (see [4, 12])
\[
\left( \sum_{m=1}^{n} z_m \right)^t \leq t \sum_{m=1}^{n} z_m \left( \sum_{k=1}^{m} z_k \right)^{t-1},
\]  
(2.7)
where \( t \geq 1 \) is constant and \( z_m \geq 0 (m \in \mathbb{N}) \), it is easy to observe that
\[
\left( \frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m v_m (c_m a_m)^p \right)^{1/p} \leq \frac{1}{p \Lambda_n^{1/p}} \sum_{m=1}^{n} \lambda_m v_m (c_m a_m)^p \left( \sum_{k=1}^{m} \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p}
\]  
(2.8)
for \( \Lambda_n \geq 1 \) and \( 0 < p \leq 1 \). Then, by (2.6) and (2.8), we obtain
\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n} \right)^{1/\Lambda_n}
\leq \frac{1}{p} \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n^{1/p}} \left( c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n} \right)^{1/\Lambda_n} \sum_{m=1}^{n} \lambda_m v_m (c_m a_m)^p \left( \sum_{k=1}^{m} \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p}
= \frac{1}{p} \sum_{m=1}^{n} \lambda_m v_m (c_m a_m)^p \left( \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n^{1/p}} \left( c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n} \right)^{1/\Lambda_n} \left( \sum_{k=1}^{m} \lambda_k v_k (c_k a_k)^p \right)^{(1-p)/p} \right).
\]  
(2.9)
Choosing $c_1^1 v_1, c_2^2 v_2, \ldots, c_n^n v_n = (\Lambda_{n-1}^{(1-1/p)} \Lambda_{n+1}^{-1})^{\Lambda_n} (n \in \mathbb{N})$ and setting $\Lambda_0 = 1$, from (2.1), it follows that

$$c_n = \left( \frac{\Lambda_{n+1}^{(1-1/p)} \Lambda_{n-1}^{1/p} v_{n+1}}{\Lambda_{n-1}^{(1-1/p)} \Lambda_{n}^{1/p} v_{n+1} \Lambda_{n}^{1/p} v_{n} \Lambda_{n}^{1/p} v_{n+1}} \right)^{\Lambda_n / \Lambda_n v_n}$$ (2.10)

This implies that

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} \left( a_1^1 v_1 a_2^2 v_2 \cdots a_n^n v_n \right)^{1/\Lambda_n}}{1 + \frac{\lambda_{n} v_{n}}{\Lambda_{n}}} \leq \frac{1}{p} \sum_{m=1}^{n} \lambda_{m} v_{m} \left( c_{m} a_{m} \right)^{p} \sum_{n=m}^{\infty} \frac{\lambda_{n+1} v_{n+1}}{\lambda_{n} \Lambda_{n+1}} \left( \sum_{k=1}^{m} \lambda_{k} v_{k} \left( c_{k} a_{k} \right)^{p} \right)^{(1-p)/p}$$ (2.11)

Hence, using the inequality [19, Lemma 3.1]

$$\left( 1 + \frac{1}{x} \right)^{x} \leq e \left( \frac{1 - \beta / x}{1 + 1/x} \right)^{\alpha}$$ (2.12)

for $x > 1$, and $\alpha, \beta$ satisfying $0 \leq \alpha \leq 1 / \ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, and $e \beta + 2^{1+\alpha} = e$, we have (2.2). Thus Theorem 2.1 is proved.

Taking $v_n = 1 (n \in \mathbb{N})$ in Theorem 2.1, we have the following corollary.

**Corollary 2.2.** Let $\lambda_n > 0$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_{m} (n \in \mathbb{N})$, $0 < p \leq 1$, and $0 < \sum_{n=1}^{\infty} \lambda_{n} a_{n} < \infty$. If

$$\frac{\Lambda_{n+1} \Lambda_{n}}{\Lambda_{n} + \lambda_{n}} \leq \Lambda_{n-1}^{(1-1/p) \Lambda_{n-1}^{1/p} \Lambda_{n}^{1/p}}$$ (2.13)
\[ \sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} \leq \frac{e^p}{p} \sum_{n=1}^{\infty} \left( \frac{\Lambda_n^{\alpha} (\Lambda_n - \lambda_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n)^\alpha} \right)^p \lambda_n a_n^p . \]  

(2.14)

**Remark 2.3.** Corollary 2.2 is a result corresponding to Theorem 1.1 with a weaker condition for \( \lambda_n (n \in \mathbb{N}) \). Further, setting \( p = 1 \) in Corollary 2.2, we obtain an inequality corresponding to inequality (1.5) with a weaker condition for \( \Lambda_n (n \in \mathbb{N}) \).

Setting \( p = 1 \) in Theorem 2.1, we obtain the following corollary.

**Corollary 2.4.** Let \( \lambda_n > 0, \nu_n > 0, a_n \geq 0, \Lambda_n = \sum_{m=1}^{n} \lambda_m \nu_m (n \in \mathbb{N}) \), and \( 0 < \sum_{n=1}^{\infty} \lambda_n \nu_n a_n < \infty \). If

\[ \frac{\Lambda_{n+1}}{\Lambda_n + \lambda_n \nu_n} \leq \nu_n^{-\frac{(\Lambda_{n-1} - \lambda_n \nu_n)}{\Lambda_n - \lambda_n \nu_n}} \nu_{n+1}, \]  

then

\[ \sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{1/\Lambda_n} \leq e \sum_{n=1}^{\infty} \left( \frac{\Lambda_n^{\alpha} (\Lambda_n - \lambda_n \nu_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n \nu_n)^\alpha} \right) \lambda_n \nu_n a_n . \]  

(2.15)

**Remark 2.5.** Inequality (2.16) is a new refinement and generalization of inequality (1.3).

**Corollary 2.6.** Under the assumptions of Theorem 2.1, if \( \alpha = 0, \beta = 1 - 2/e \), then

\[ \sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{1/\Lambda_n} \leq \frac{e^p}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{\lambda_n \nu_n (1 - 2/e)}{\Lambda_n} \right)^p \lambda_n \nu_n a_n \Lambda_n^{p-1} \left( \sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p \right)^{(1-p)/p} , \]  

where (2.3) holds.

**Remark 2.7.** Corollary 2.6 is a new refinement and generalization of [10, Corollary 2.3]. Further, setting \( \nu_n = 1 (n \in \mathbb{N}) \) in Corollary 2.6, we obtain a result corresponding to [10, Corollary 2.3] with a weaker condition for \( \lambda_n (n \in \mathbb{N}) \).

**Corollary 2.8.** Under the assumptions of Theorem 2.1, if \( \alpha = 1/\ln 2 - 1, \beta = 0 \), then

\[ \sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{1/\Lambda_n} \leq \frac{e^p}{p} \sum_{n=1}^{\infty} \left( 1 + \frac{\lambda_n \nu_n}{\Lambda_n} \right)^{1 - 1/\ln 2} \lambda_n \nu_n a_n \Lambda_n^{p-1} \left( \sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p \right)^{(1-p)/p} , \]  

where (2.3) holds.
Remark 2.9. Corollary 2.8 is a new refinement and generalization of [10, Corollary 2.4]. Further, setting $\nu_n = 1$ ($n \in \mathbb{N}$) in Corollary 2.8, we obtain a result corresponding to [10, Corollary 2.4] with a weaker condition for $\lambda_n$ ($n \in \mathbb{N}$).

A new general refined Hardy’s inequality is introduced to the following theorem.

**Theorem 2.10.** Let $\lambda_n > 0$, $\nu_n > 0$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m \nu_m$ ($n \in \mathbb{N}$), and $0 < \sum_{n=1}^{\infty} \lambda_n \nu_n a_n < \infty$ for $0 < p \leq t < \infty$. If

$$\frac{\Lambda_{n+1} \Lambda_n}{\Lambda_n + \lambda_n \nu_n} \leq \Lambda_{n-1}^{((p-t)/p)/(\Lambda_{n-1}/\Lambda_n)} \Lambda_n^{t/p} \nu_n^{-(\Lambda_{n-1}/\Lambda_n)} \nu_{n+1},$$

then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{t/\Lambda_n} \leq \frac{t e^{p/t}}{p} \sum_{n=1}^{\infty} \left( \frac{\Lambda_n^{(1-p)/p} \Lambda_{n+1}^{p/(1-p)}}{\Lambda_n (\lambda_n \nu_n)^{1/p}} \right)^{p/t} \lambda_n \nu_n a_n^{p/(1-t)} \left( \sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p \right)^{(t-p)/p},$$

(2.20)

where

$$c_k = \left[ \frac{\Lambda_k^{(1-t/p) \lambda_{k+1}} \Lambda_{k+1}^{(1-t/p) \lambda_k}}{\Lambda_k^{(1-t/p) \lambda_{k+1}} \lambda_k^{1/\lambda_k} \nu_{k+1}^{1/\lambda_k}} \right],$$

(2.21)

and $\alpha, \beta$ satisfy $0 \leq \alpha \leq 1/\ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, and $e \beta + 2^{1+\alpha} = e$.

**Proof.** The proof is almost the same as in Theorem 2.1. By the power mean inequality [11, page 15], we have (2.4) for $\alpha_m \geq 0$, $p > 0$, and $q_m > 0$ ($m \in \mathbb{N}$) with $\sum_{m=1}^{n} q_m = 1$. By (2.4), we obtain

$$\left( \alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n} \right)^t \leq \left( \sum_{m=1}^{n} q_m \alpha_m^p \right)^{t/p},$$

(2.22)

for $t > 0$. Taking $c_m > 0$, $\alpha_m = c_m a_m$, and $q_m = \lambda_m \nu_m / \Lambda_n$, we have

$$\left( (c_1 a_1)^{\lambda_1 \nu_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 \nu_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n \nu_n / \Lambda_n} \right)^t \leq \left( \sum_{m=1}^{n} \frac{\lambda_m \nu_m}{\Lambda_n} (c_m a_m)^p \right)^{t/p}.$$

(2.23)
Using the above inequality, we have

\[ \sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n} \right)^{t/\Lambda_n} = \sum_{n=1}^{\infty} \lambda_{n+1} \left( \frac{c_1 a_1^{\lambda_1 v_1} (c_2 a_2)^{\lambda_2 v_2} \cdots (c_n a_n)^{\lambda_n v_n}}{c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n}} \right)^{t/\Lambda_n} \leq \sum_{n=1}^{\infty} \left( \frac{\lambda_{n+1}}{c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n}} \right)^{t/\Lambda_n} \left( \sum_{m=1}^{n} \frac{\lambda_m v_m (c_m a_m)^p}{\Lambda_n} \right)^{t/p}. \] (2.24)

By using the inequality (see [4, 12])

\[ \left( \sum_{m=1}^{n} z_m \right)^t \leq t \sum_{n=1}^{\infty} z_m \left( \sum_{k=1}^{m} z_k \right)^{t-1}, \] (2.25)

where \( t \geq 1 \) is constant and \( z_m > 0 \ (n \in \mathbb{N}) \), it is easy to observe that

\[ \frac{1}{\Lambda_{n}^{t/p}} \left( \sum_{m=1}^{n} \frac{\lambda_m v_m (c_m a_m)^p}{\Lambda_n} \right)^{t/p} \leq \frac{t}{p \Lambda_n^{t/p}} \sum_{m=1}^{n} \frac{\lambda_m v_m (c_m a_m)^p}{\Lambda_n} \left( \sum_{k=1}^{m} \lambda_k v_k (c_k a_k)^p \right)^{(t-p)/p}. \] (2.26)

for \( 0 < p \leq t \). Then, by (2.24) and (2.26), we obtain

\[ \sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 v_1} a_2^{\lambda_2 v_2} \cdots a_n^{\lambda_n v_n} \right)^{t/\Lambda_n} \leq \frac{t}{p \sum_{n=1}^{\infty} \lambda_{n+1}^{t/p} \left( c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n} \right)^{t/\Lambda_n}} \sum_{m=1}^{n} \frac{\lambda_m v_m (c_m a_m)^p}{\Lambda_n} \left( \sum_{k=1}^{m} \lambda_k v_k (c_k a_k)^p \right)^{(t-p)/p}. \] (2.27)

Choosing \( c_1^{\lambda_1 v_1} c_2^{\lambda_2 v_2} \cdots c_n^{\lambda_n v_n} = (\Lambda_n(1-t/p) \Lambda_{n+1} v_{n+1})^{\lambda_n/t} \ (n \in \mathbb{N}) \) and setting \( \Lambda_0 = 1 \), from (2.19), it follows that

\[ c_n = \left( \frac{(\Lambda_n(1-t/p) \Lambda_{n+1} v_{n+1}^{-1})^{\lambda_n/t}}{(\Lambda_n(1-t/p) \Lambda_{n+1} v_{n+1}^{-1})^{\lambda_n-1/t}} \right)^{1/\lambda_n v_n} = \left( \frac{(\Lambda_{n+1} v_{n+1})^{\lambda_n/v_n}}{(\Lambda_n v_n)^{\lambda_n/v_n}} \right)^{\lambda_n v_n/t} \frac{\Lambda_n}{\Lambda_n(v_n^{\lambda_n/v_n})} \frac{1}{\Lambda_n^{1/t}}. \] (2.28)
This implies that
\[
\sum_{n=1}^{\infty} \left( a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{t/\Lambda_n} \leq \frac{t}{p} \sum_{m=1}^{n} \lambda_m \nu_m (c_m a_m)^p \sum_{n=m}^{\infty} \lambda_{n+1} \nu_{n+1} \left( \sum_{k=1}^{m} \lambda_k \nu_k (c_k a_k)^p \right)^{(t-p)/p}.
\]

Hence, using inequality (2.12) [19, Lemma 3.1] for \( x > 1 \), and \( \alpha, \beta \) satisfy \( 0 \leq \alpha \leq 1/\ln 2 - 1, 0 \leq \beta \leq 1 - 2/e \), and \( e\beta + 2^{1+\alpha} = e \), we have (2.20). Thus Theorem 2.10 is proved.

**Remark 2.11.** Theorem 2.10 reduces to Theorem 2.1 when \( t = 1 \). Hence, inequality (2.20) is a new generalization of Hardy’s inequality. Taking \( \nu_n = 1 \) \( (n \in \mathbb{N}) \) in Theorem 2.10, we obtain a result corresponding to [10, Theorem 2.6] with a weak condition for \( \lambda_n \) \( (n \in \mathbb{N}) \). Also assuming that \( \lambda_n = 1 \) in Theorem 2.10, we have an extension of [10, Corollary 2.7] as in the following corollary.

**Corollary 2.12.** Let \( \nu_n > 0, a_n \geq 0, \Lambda_n = \sum_{m=1}^{n} \nu_m \) \( (n \in \mathbb{N}) \), and \( 0 < \sum_{n=1}^{\infty} \nu_n a_n < \infty \) for \( 0 < p \leq t < \infty \). If
\[
\frac{\Lambda_{n+1} \Lambda_n}{\Lambda_n + \nu_n} \leq \Lambda_{n-1}^{(p-t)/p} \Lambda_n^{(p-t)/p} \nu_{n-1}^{(p-t)/p} \nu_n^{p/t} a_n^{p/(t-p)} \sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p \right)^{(t-p)/p}.
\]

then
\[
\sum_{n=1}^{\infty} \left( a_1^{\nu_1} a_2^{\nu_2} \cdots a_n^{\nu_n} \right)^{t/\Lambda_n} \leq \frac{t e^{p/t}}{p} \sum_{n=1}^{\infty} \left( \frac{\Lambda_n \nu_n}{\Lambda_n + \nu_n} \right)^{p/t} \nu_n a_n^{p/(t-p)} \left( \sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p \right)^{(t-p)/p},
\]

where
\[
c_k = \left( \frac{\Lambda_k^{(1-t)/p} \Lambda_{k+1}^{1-\nu_k}}{\Lambda_k^{(1-t)/p} \Lambda_{k+1}^{1-\nu_k}} \right)^{1/\nu_k t}.
\]
Corollary 2.13. Under the assumptions of Theorem 2.10, if $\alpha = 0$, $\beta = 1 - 2/e$, then
\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{t/\Lambda_n} \leq \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{\lambda_n \nu_n (1 - 2/e)}{\Lambda_n} \right)^{p/t} \lambda_n \nu_n a_n^{p} \Lambda_n^{(p-t)/t} \left( \sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p \right)^{ (t-p)/p },
\]
(2.33)
where
\[
c_k^{\lambda_k \nu_k t} = \frac{\left( \Lambda_k^{(1-t/p) \Lambda_{k+1}} \right) \Lambda_k^{\Lambda_k - 1} \nu_k^{\Lambda_k - 1} (\Lambda_{k-1}^{(1-t/p)} \Lambda_k^{\Lambda_k - 1} \nu_k^{\Lambda_k - 1})}{\Lambda_k^{\Lambda_k - 1} \nu_k^{\Lambda_k - 1} (\Lambda_{k-1}^{(1-t/p)} \Lambda_k^{\Lambda_k - 1} \nu_k^{\Lambda_k - 1})}. \tag{2.34}
\]

Corollary 2.14. Under the assumptions of Theorem 2.1, if $\alpha = 1/\ln 2 - 1$, $\beta = 0$, then
\[
\sum_{n=1}^{\infty} \lambda_{n+1} \left( a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{t/\Lambda_n} \leq \frac{te^{p/t}}{p} \sum_{n=1}^{\infty} \left( 1 - \frac{\lambda_n \nu_n}{\Lambda_n} \right)^{1-1/\ln 2} \lambda_n \nu_n a_n^{p} \Lambda_n^{(p-t)/t} \left( \sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p \right)^{ (t-p)/p },
\]
(2.35)
where (2.34) holds.

Remark 2.15. Corollaries 2.13 and 2.14 are new refinements and generalizations of [10, Corollaries 2.8 and 2.9], respectively. Further, taking $\nu_n = 1$ ($n \in \mathbb{N}$) in Corollaries 2.13 and 2.14, we obtain results corresponding to [10, Corollaries 2.8 and 2.9] with a weaker condition for $\lambda_n$ ($n \in \mathbb{N}$).

References


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