THE CLASS OF FUNCTIONS SPIRALLIKE WITH RESPECT TO A BOUNDARY POINT

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The aim of this paper is to present an analytic characterization of the class of functions δ-spirallike with respect to a boundary point. The method of proof is based on Julia’s lemma.

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1. Introduction. In this paper, we study the class \( \mathcal{S}_0^\ast (\delta) \) of \( \delta \)-spirallike functions with respect to a boundary point. Spirallikeness with respect to a boundary point is a fresh idea being the subject of studies in [1, 2]. Cited papers developed the method based, on the one hand, on the analytic formula for the class \( \mathcal{S}_0^\ast \) of functions starlike with respect to a boundary point proposed and proved partially by Robertson [10], and, on the other hand, on some dynamical system built for \( \mathcal{S}_0^\ast \). Lyzzaik [8] completing Robertson’s proof solved positively his conjecture. Thereby the full analytic description of functions in \( \mathcal{S}_0^\ast \) was finished. The author [5], by using the Julia lemma, proposed an alternative analytic formula for the class \( \mathcal{S}_0^\ast \) different than Robertson’s characterization. The necessary condition for functions to be in \( \mathcal{S}_0^\ast \) was shown and, partially, the sufficient condition. In [7], Lyzzaik and the author complete the proof and in this way the class \( \mathcal{S}_0^\ast \) was equipped with a new analytic characterization.

The use of the Julia lemma has the virtue of looking at the inner property of the class \( \mathcal{S}_0^\ast \) and the other classes defined by the geometric property connected with the boundary point (see, e.g., [6]). In this paper, we apply once again the Julia lemma as a technique to study the class \( \mathcal{S}_0^\ast (\delta) \). Theorem 3.5 demonstrates the basic observation that spirallikeness, as earlier starlikeness with respect to a boundary point, is preserved on each oricycle in the unit disk by every function in \( \mathcal{S}_0^\ast (\delta) \). Theorems 3.6 and 3.8 complete a new analytic characterization of \( \delta \)-spirallike functions with respect to a boundary point.

2. Preliminaries. Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( \mathbb{T} = \partial \mathbb{D} \). For each \( k > 0 \), consider the oricycle

\[
\mathcal{O}_k = \left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} < k \right\}.
\]

The oricycle \( \mathcal{O}_k \) is a disk in \( \mathbb{D} \) whose boundary circle \( \partial \mathcal{O}_k \) is tangent to \( \mathbb{T} \) at 1.

Let \( \Delta = \{ z \in \mathbb{D} : |\arg(1-z)| < b, |z-1| < \rho \}, b \in (0, \pi/2), \rho < 2\cos b \), be a Stoltz angle at 1.
Let \( \mathcal{A} \) denote the set of all analytic functions in \( \mathbb{D} \). The subset of \( \mathcal{A} \) of all univalent functions will be denoted by \( \mathcal{S} \). The set of all \( \omega \in \mathcal{A} \) such that \(|\omega(z)| < 1\) for \( z \in \mathbb{D} \) will be denoted by \( \mathcal{B} \).

An angular limit of \( f \in \mathcal{A} \) at \( \zeta \in \mathbb{T} \) will be denoted by \( f_\angle(\zeta) \). An angular derivative of \( f \in \mathcal{A} \) at \( \zeta \in \mathbb{T} \) will be denoted by \( f_\prime_\angle(\zeta) \).

Let \( f \in \mathcal{A} \). Assume that there exists a finite radial limit \( \lim_{r \to 1^-} f(r) = \upsilon \) at 1. Denote by

\[
Q(z) = \frac{(z - 1)f'(z)}{f(z) - \upsilon}, \quad z \in \mathbb{D},
\]

the Visser-Ostrowski quotient of \( f \) at 1 (see, e.g., [9, page 251]). We say that \( f \) satisfies the Visser-Ostrowski condition at 1 if \( Q_\angle(1) = 1 \) (see, e.g., [9, page 252]).

We recall now the Julia lemma (see [4]; see also [11, pages 68–72]).

**Lemma 2.1** (Julia). Let \( \omega \in \mathcal{B} \). Assume that there exists a sequence \( (z_n) \) of points in \( \mathbb{D} \) such that

\[
\lim_{n \to \infty} z_n = 1, \quad \lim_{n \to \infty} \omega(z_n) = 1,
\]

\[
\lim_{n \to \infty} \frac{1 - |\omega(z_n)|}{1 - |z_n|} = \lambda < \infty.
\]

Then

\[
\frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} \leq \lambda \frac{|1 - z|^2}{1 - |z|^2}, \quad z \in \mathbb{D},
\]

that is, for every \( k > 0 \),

\[
\omega(\Omega_k) \subset \Omega_{\lambda k}.
\]

**Remark 2.2.** The constant \( \lambda \) defined in (2.4) is positive (see [11, pages 68–69]).

For \( \omega \in \mathcal{B} \) with \( \omega_\angle(1) = 1 \), let

\[
\Lambda = \sup \left\{ \frac{|1 - \omega(z)|^2}{1 - |\omega(z)|^2} : \frac{1 - |z|^2}{|1 - z|^2} : z \in \mathbb{D} \right\}.
\]

The next lemma is a converse of the Julia lemma (see [11, page 72] and [3, pages 42–44]).

**Lemma 2.3.** Let \( \omega \in \mathcal{B} \). If (2.5) holds for some \( \lambda > 0 \), then there exists a sequence \( (z_n) \) of points in \( \mathbb{D} \) satisfying (2.3) and such that

\[
\lim_{n \to \infty} \frac{1 - |\omega(z_n)|}{1 - |z_n|} = \Lambda \leq \lambda.
\]
Remark 2.4. In fact, in Lemma 2.3, we can find a sequence of real numbers \((x_n)\) in \((0,1)\) satisfying (2.3) and (2.8). Also it can be proved that then \(\omega_\angle(1) = 1\) and

\[
\lim_{\Delta \ni z \rightarrow 1} \frac{1 - \omega(z)}{1 - z} = \lim_{\Delta \ni z \rightarrow 1} \omega'(z) = \omega'_\angle(1) = \Lambda
\]

for every Stolz angle \(\Delta\) (see [3, pages 42–44]).

For our need, it will be convenient to define the following classes of functions introduced in [5].

Definition 2.5. Fix \(\lambda \in (0, \infty]\). \(\omega \in \mathbb{B}\) is said to belong to the class \(\mathcal{B}(\lambda)\) if \(\omega_\angle(1) = 1\) and \(\omega'_\angle(1) = \lambda\).

Let \(\mathcal{B}(\lambda)\) denote the class of all functions \(p\) of the form

\[
p(z) = 4 \frac{1 - \omega(z)}{1 + \omega(z)}, \quad z \in \mathbb{D},
\]

where \(\omega \in \mathcal{B}(\lambda)\).

Remark 2.6. Note that \(p \in \mathcal{B}(\lambda)\) if and only if \(p_\angle(1) = 0\) and \(p'_\angle(1) = -2\lambda\).

3. Spirallikeness with respect to a boundary point.

3.1. Let for \(w \in \mathbb{C}\) and \(A \subset \mathbb{C}\), \(wA = \{wu : u \in A\}\).

We start with the following definition.

Definition 3.1. Fix \(\delta \in (-\pi/2, \pi/2)\) and let \(L(\delta) = \{\exp(e^{-i\delta}t) : t \leq 0\}\) be the logarithmic spiral joint 0 and 1. Clearly, \(L(0)\) is a line segment \((0,1]\). Let \(\mathcal{F}_0^*(\delta)\) denote the class of simply connected domains \(\Omega \subset \mathbb{C}\) with \(0 \in \partial \Omega\) and such that \(wL(\delta) \subset \Omega\) for every \(w \in \Omega\). Let \(\mathcal{F}_0^*(\delta) \subset \mathcal{F}\) denote the corresponding class of functions mapping \(\mathbb{D}\) onto domains in \(\mathcal{F}_0^*(\delta)\).

Domains in \(\mathcal{F}_0^*(\delta)\) and functions in \(\mathcal{F}_0^*(\delta)\) will be called \(\delta\)-spirallike with respect to the boundary point at the origin.

For \(\delta = 0\), we get the class \(\mathcal{F}_0^*(0)\), that is, \(\mathcal{F}_0^* = \mathcal{F}_0^*(0)\). Recall that \(f\) belongs to \(\mathcal{F}_0^*\) if and only if it is univalent in \(\mathbb{D}\) and \(f(\mathbb{D})\) is a starlike domain with respect to the boundary point at the origin, that is, the line segment \((0,w]\) is a subset of \(f(\mathbb{D})\) for every \(w \in f(\mathbb{D})\) (for more about the class \(\mathcal{F}_0^*\), see [5, 7, 8, 10]).

Let \(f \in \mathcal{F}_0^*(\delta)\) for \(\delta \in (-\pi/2, \pi/2)\), and fix \(w_1 \in f(\mathbb{D})\). Then \(w_1L(\delta) \subset \Omega\) is a curve ending at the origin, so by [9, Proposition 2.1, page 29], the preimage of \(w_1L(\delta)\) is a curve in \(\mathbb{D}\) ending at some point \(\zeta_0\) of \(\mathbb{T}\). Applying [9, Corollary 2.17, page 35], we conclude that \(f\) has the angular limit zero at \(\zeta_0\).

Proposition 3.2. Every function \(f \in \mathcal{F}_0^*(\delta)\), \(\delta \in (-\pi/2, \pi/2)\), has the angular limit zero at some point \(\zeta_0 \in \mathbb{T}\), that is, \(f_\angle(\zeta_0) = 0\).

In the following considerations, we assume that \(\zeta_0 = 1\), that is, we use the boundary normalization \(f_\angle(1) = 0\).
3.2. In the proofs of the main theorems of this paper, we will need two lemmas proved in [5].

**Lemma 3.3.** Every sequence \((a_n)\) of positive numbers with
\[
\lim_{n \to \infty} (a_1 a_2 \cdots a_n) = 0 \tag{3.1}
\]
has a convergent subsequence \((a_{n_k})\) and
\[
0 \leq \lim_{k \to \infty} a_{n_k} = a \leq 1. \tag{3.2}
\]

**Lemma 3.4.** Let \(f \in \mathcal{H}\) have a radial limit \(\lim_{r \to 1^-} f(r) = \nu\). Then there exist \(\lambda \in [0, 1]\) and a sequence \((r_n)\) with \(0 < r_n < 1\) and \(\lim_{n \to \infty} r_n = 1\) such that
\[
\lim_{n \to \infty} |Q(r_n)| = 2\lambda. \tag{3.3}
\]

3.3. The theorem below says that every function in \(\mathcal{H}_0^*(\delta)\) having a boundary normalization \(f_\mathcal{E}(1) = 0\) preserves spirallikeness with respect to a boundary point on each oricycle in \(\mathbb{D}\). This information will be used later to find an analytic formula for functions in \(\mathcal{H}_0^*(\delta)\).

**Theorem 3.5.** Fix \(\delta \in (-\pi/2, \pi/2)\) and let \(f \in \mathcal{H}\). Then \(f \in \mathcal{H}_0^*(\delta)\) and \(f_\mathcal{E}(1) = 0\) if and only if \(f(\mathbb{D}_k) \in \mathcal{H}_0^*(\delta)\) for every \(k > 0\).

**Proof.** Assume that \(f \in \mathcal{H}_0^*(\delta)\) and \(f_\mathcal{E}(1) = 0\). For each \(t \leq 0\), define
\[
\omega_t(z) = f^{-1}(\exp(e^{-i\delta}t)f(z)), \quad z \in \mathbb{D}. \tag{3.4}
\]
Since \(f(\mathbb{D}) \in \mathcal{H}_0^*(\delta)\), \(\exp(e^{-i\delta}t)f(z) \in f(\mathbb{D})\) for every \(t \leq 0\), \(z \in \mathbb{D}\), and the univalence of \(f\) shows that \(\omega_t\) is well defined for each \(t \leq 0\).

Now, fix \(t < 0\) and \(w_1 \in f(\mathbb{D})\). Hence \(w_1L(\delta) \subset f(\mathbb{D})\). For \(n \in \mathbb{N}\), let
\[
w_n = \exp(e^{-i\delta}(n-1)t)w_1 \tag{3.5}
\]
and \(z_n = f^{-1}(w_n)\). Since the sequence \((w_n)\) is placed on the logarithmic spiral \(w_1L(\delta)\) and \(\lim_{n \to \infty} w_n = 0\), \(\lim_{n \to \infty} z_n = 1\) by Proposition 3.2. Observe that
\[
\omega_t(z_n) = f^{-1}(\exp(e^{-i\delta}t)w_n) = f^{-1}(\exp(e^{-i\delta}nt)w_1) = z_{n+1}. \tag{3.6}
\]
Let now
\[
a_n = \frac{1 - |\omega_t(z_n)|}{1 - |z_n|}, \quad n \in \mathbb{N}. \tag{3.7}
\]
Hence
\[
a_n = \frac{1 - |z_{n+1}|}{1 - |z_n|} \tag{3.8}
\]
for all \( n \in \mathbb{N} \). Consequently,

\[
\lim_{n \to \infty} (a_1a_2 \cdots a_n) = \lim_{n \to \infty} \left( \frac{1 - |z_2|}{1 - |z_1|} \frac{1 - |z_3|}{1 - |z_2|} \cdots \frac{1 - |z_n|}{1 - |z_{n-1}|} \frac{1 - |z_{n+1}|}{1 - |z_n|} \right) \tag{3.9}
\]

which means that

\[
0 \leq \lim_{k \to \infty} a_{n_k} = \lambda(t) \leq 1,
\]

which means that

\[
\lim_{k \to \infty} \frac{1 - |\omega_t(z_{n_k})|}{1 - |z_{n_k}|} = \lambda(t) \leq 1 \tag{3.11}
\]

for each \( t < 0 \). In view of Remark 2.2, \( \lambda(t) > 0 \) for every \( t < 0 \).

Hence, each \( \omega_t \) satisfies the assumptions of the Julia lemma, and since \( \lambda(t) \leq 1 \) for every \( t < 0 \), we derive that \( \omega_t(\mathbb{O}_k) \subset \mathbb{O}_{\lambda(t)k} \subset \mathbb{O}_k \) for every \( k > 0 \). This yields \( \exp(e^{-i\delta}t)f(\mathbb{O}_k) \subset f(\mathbb{O}_k) \) for every \( t < 0 \), and hence \( f(\mathbb{O}_k) \in \mathcal{F}^+_\delta(\delta) \).

Conversely, assume that \( f(\mathbb{O}_k) \in \mathcal{F}^+_\delta(\delta) \) for every \( k > 0 \). Since \( 0 \in \cap_{k > 0} \partial f(\mathbb{O}_k) \) and

\[
f(\mathbb{D}) = \bigcup_{k > 0} f(\mathbb{O}_k),
\]

it follows that \( 0 \in \partial f(\mathbb{D}) \) and \( f(\mathbb{D}) \in \mathcal{F}^+_\delta(\Delta) \), so \( f \in \mathcal{F}^+_\delta(\delta) \). We show that \( f_\varepsilon(1) = 0 \). Fix \( k > 0 \) and \( w_1 \in f(\mathbb{O}_k) \). Then \( w_1 L(\delta) \subset f(\mathbb{O}_k) \) is a curve ending at \( 0 \in \partial f(\mathbb{D}) \). By [9, Proposition 2.14, page 29], \( f^{-1}(w_1 L(\delta)) \) is a curve in \( \mathbb{D} \) ending at some point \( \zeta_0 \) of \( \mathbb{T} \). Since \( f^{-1}(w_1 L(\delta)) \subset \mathbb{O}_k \) and \( \partial \mathbb{O}_k \cap \mathbb{T} = \{1\} \), we have \( \zeta_0 = 1 \). The proof of the theorem is finished. \( \square \)

Using Theorem 3.5, we characterize functions in \( \mathcal{F}^+_\delta(\delta) \) as follows.

**Theorem 3.6.** Fix \( \delta \in (-\pi/2, \pi/2) \). If \( f \in \mathcal{F}^+_\delta(\delta) \) and \( f_\varepsilon(1) = 0 \), then there exist \( \lambda \in (0,1] \) and \( \omega \in \mathcal{B}(\lambda) \) such that

\[
-e^{i\delta}(1-z)^2\frac{f''(z)}{f(z)} = 4\frac{1-\omega(z)}{1+\omega(z)}, \quad z \in \mathbb{D}. \tag{3.13}
\]

**Proof.** The case \( \delta = 0 \) reduces to [5, Theorem 3.1]. Therefore we assume that \( \delta \in (-\pi/2, \pi/2) \setminus \{0\} \). Let \( O_k = \partial \mathbb{O}_k \setminus \{1\} \) and \( \Gamma_k = \partial f(\mathbb{O}_k) \) for every \( k > 0 \). First, we show that

\[
\text{Re} \left\{ e^{i\delta}(1-z)^2\frac{f''(z)}{f(z)} \right\} < 0, \quad z \in \mathbb{D}. \tag{3.14}
\]

We prove that the last inequality is true for all points on \( O_k \) for every \( k > 0 \). Now, fix \( k > 0, z \in O_k \) and let \( w = f(z) \). We parametrize \( O_k \) as follows:

\[
O_k : z = y_k(\theta) = \frac{1+k e^{i\theta}}{1+k}, \quad \theta \in (0,2\pi). \tag{3.15}
\]
Thus \( O_k \) is positively oriented. Denote by \( \tau(z) \) the tangent vector to \( \Gamma_k \) at \( w = f(z) \), that is, \( \tau(z) = y_k'(\theta) f'(y_k(\theta)) \), where \( z = y_k(\theta) \). Since

\[
(1 - y_k'(\theta))^2 = \frac{k^2}{(1 + k)^2} (1 - e^{i\theta})^2 = -\frac{4k \sin^2(\theta/2)}{(k + 1)i} \left( \frac{k}{k + 1} e^{i\theta}i \right),
\]

(3.16)

we have

\[
\tau(z) = \frac{-i(1 - y_k(\theta))^2 f'(y_k(\theta))}{2 \text{Re} \{1 - y_k(\theta)\}} = \frac{-i(1 - z)^2 f'(z)}{2 \text{Re} \{1 - z\}}.
\]

(3.17)

Let

\[
w(t) = f(z) \exp(e^{-i\delta} t), \quad t \leq 0,
\]

(3.18)

be a parametrization of \( f(z) L(\delta) \) and let \( w'(0) = \lim_{t \to 0^-} w'(t) = e^{-i\delta} f(z) \) be the one-sided tangent vector to the logarithmic spiral \( f(z) L(\delta) \) at \( f(z) \). By \( \varphi(z) \) we denote the directed angle from the vector \( i w'(0) \) to \( \tau(z) \), that is,

\[
\varphi(z) = \arg \{ \tau(z) \} - \arg \{ i w'(0) \} = \arg \left\{ -\frac{i(1 - z)^2 f'(z)}{2 \text{Re} \{1 - z\}} \right\} - \arg \{ i e^{-i\delta} f(z) \} = \arg \left\{ -e^{i\delta} \frac{(1 - z)^2 f'(z)}{f(z)} \right\}.
\]

(3.19)

By Theorem 3.5, \( f(O_k) \in \mathcal{H}^+(\delta) \) for every \( k > 0 \). Hence it is easy to see that

\[
wL(\delta) \subset \overline{f(O_k)},
\]

(3.20)

where \( w = f(z) \in \Gamma_k \). Indeed, let \( w_0 \in wL(\delta) \) be arbitrary. Thus \( w_0 = wu_0 \) for some \( u_0 \in L(\delta) \). Since \( w \in \Gamma_k \), there exists a sequence \( (w_n) \) of points in \( f(O_k) \) convergent to \( w \). The inclusion \( w_n L(\delta) \subset f(O_k) \) yields that \( w_n u_0 \) is a point of \( f(O_k) \) for every \( n \in \mathbb{N} \). At the end, the convergence of the sequence \( (w_n u_0) \) of points of \( f(O_k) \) to \( w_0 \) implies that \( w_0 \in f(O_k) \). Since \( w_0 \) was arbitrary, our claim is proved.

Let \( l \) be a line going through \( f(z) \) with \( w'(0) \) as the directional vector. Then \( l \) divides the plane into two closed half-planes \( H_1 \) and \( H_2 \). One of them, say \( H_1 \), contains the origin and the spiral \( f(z) L(\delta) \). We assume first that \( \delta \in (-\pi/2, 0) \). This means that the spiral \( L(\delta) \) has the shape such that it attains 1 from the lower half-plane. Moreover, \( f(z) L(\delta) \) parametrized as above turns round the origin in the counterclockwise direction. Hence, \( i w'(0) \) lies in \( H_1 \). By Theorem 3.5, \( f(O_k) \in \mathcal{H}^+(\delta) \). Hence, and from (3.20), it follows that either \( \Gamma_k \) is tangent both to \( f(z) L(\delta) \) (one-sided) and to \( l \) at \( f(z) \), and then \( \tau(z) \) lies in \( l \) so in \( H_1 \), or by [9, Proposition 2.13, page 28], there is a crosscut \( C \subset l \) of \( f(O_k) \) with one endpoint at \( f(z) \). Thus, by [9, Proposition 2.12, page 27], \( f(O_k) \) has exactly two components, one of them, say \( G \), lies in \( H_2 \). Clearly, \( \partial G = C \cup \Gamma \), where \( \Gamma \subset \Gamma_k \) ends at \( f(z) \). Hence \( \Gamma \) is a subset of \( H_2 \) and, since it is part of a positively oriented closed
analytic curve $\Gamma_k$, we deduce finally that the tangent vector $\tau(z)$ to $\Gamma_k$ at $f(z)$ lies in $H_1$. In a similar way, we can prove that both vectors $iw'(0)$ and $\tau(z)$ lie together in $H_2$ as $\delta \in (0,\pi/2)$. This, (3.19), and the fact that $iw'(0)$ is orthogonal to $l$ yield

$$|\varphi(z)| \leq \frac{\pi}{2}. \quad (3.21)$$

As $k > 0$ and $z \in O_k$ was arbitrary, this is true in $\mathbb{D}$.

Suppose now that equality holds in (3.21) for some $z_0 \in \mathbb{D}$. By the maximum principle for harmonic functions, it holds in the whole disk $\mathbb{D}$, which implies that there exists $\gamma \in \mathbb{R} \setminus \{0\}$ so that

$$e^{i\delta}(1-z^2)f'(z) = \gamma i, \quad z \in \mathbb{D}. \quad (3.22)$$

But the solution

$$f(z) = f_0(z) = f(0) \exp \left( \frac{e^{-i\delta} \gamma i z}{1-z} \right), \quad z \in \mathbb{D}, \quad (3.23)$$

of the last equation is not univalent in $\mathbb{D}$. So $f_0 \not\in \mathcal{G}_0^+(\delta)$, and hence strict inequality holds in (3.21).

Let $p(z) = -e^{i\delta}(1-z^2)f'(z)/f(z)$ and let

$$\omega(z) = \frac{4-p(z)}{4+p(z)}, \quad z \in \mathbb{D}. \quad (3.24)$$

Then $\omega(\mathbb{D}) \subset \mathbb{D}$. We now prove that $\omega \in \mathcal{B}(\lambda)$ for some $\lambda \in (0,1]$. Recalling the Visser-Ostrowski quotient, we can write

$$p(z) = e^{i\delta}(1-z)Q(z), \quad z \in \mathbb{D}. \quad (3.25)$$

Since, for every $r \in (0,1)$,

$$|Q(r)| \leq \frac{4}{1+r} \quad (3.26)$$

(see [5, Lemma 2.2, (2.3)]), we have

$$\lim_{r \to 1^-} \{(1-r)Q(r)\} = \lim_{r \to 1^-} \{e^{-i\delta}p(r)\} = 0. \quad (3.27)$$

Hence $\lim_{r \to 1^-} p(r) = 0$ and, in view of (3.24), $\lim_{r \to 1^-} \omega(r) = 1$, so condition (2.3) of the Julia lemma is satisfied. By Lemma 3.4, there exist $\lambda_0 \in [0,1]$ and a sequence $(r_n)$ in $(0,1)$ with $\lim_{n \to \infty} r_n = 1$ such that

$$\lim_{n \to \infty} |Q(r_n)| = 2\lambda_0. \quad (3.28)$$
From (3.24) and (3.27) we have
\[
\lim_{n \to \infty} \frac{|1 - \omega(r_n)|}{1 - r_n} = \lim_{n \to \infty} \left\{ \frac{2}{|4 + p(r_n)|} \cdot \frac{|p(r_n)|}{1 - r_n} \right\} = \lim_{n \to \infty} \left\{ \frac{2}{|4 + p(r_n)|} |Q(r_n)| \right\} = \lambda_0 \in [0,1].
\]

But
\[
\frac{1 - |\omega(r_n)|}{1 - r_n} \leq \frac{1 - |\omega(r_n)|}{1 - r_n},
\]
so we can find a subsequence \((r_{n_k})\) of \((r_n)\) such that
\[
\lim_{k \to \infty} \frac{1 - |\omega(r_{n_k})|}{1 - r_{n_k}} = \lambda_1 \leq \lambda_0.
\]

By Remark 2.2, \(\lambda_1 \in (0,1].\) Hence \(\omega\) satisfies the assumptions of the Julia lemma with \(\lambda = \lambda_1.\) Since then (2.5) holds, by using Lemma 2.3 and Remark 2.4, we see that \(\omega \in \mathcal{B}(\Lambda),\) where \(\Lambda \leq \lambda_1 \leq 1\) is given by (2.7). This ends the proof of the theorem. □

**Corollary 3.7.** If \(f \in \mathcal{F}_0^*(\delta), \delta \in (-\pi/2, \pi/2),\) and \(f(z) = 0,\) then there exists \(\lambda \in (0,1]\) such that
\[
\lim_{\Delta \ni z \to 1} Q(z) = 2\lambda e^{-i\delta}
\]
for every Stolz angle \(\Delta.\)

**Proof.** Since
\[
Q(z) = e^{-i\delta} \frac{4}{1 + \omega(z)} \frac{1 - \omega(z)}{1 - z}, \quad z \in \mathbb{D},
\]
and
\[
\lim_{\Delta \ni z \to 1} \frac{1 - \omega(z)}{1 - z} = \Lambda \in (0,1]\]
for every Stolz angle \(\Delta,\) the assertion follows at once with \(\lambda = \Lambda.\) □

**Theorem 3.8.** Fix \(\delta \in (-\pi/2, \pi/2).\) Let \(f \in \mathcal{F} \) with \(f_z(1) = 0.\) If there exist \(\lambda \in (0,\cos \delta]\) and a function \(\omega \in \mathcal{B}(\Lambda)\) such that (3.13) holds, then \(f \in \mathcal{F}_0^*.\)

**Proof.** First we show that \(f\) is univalent in \(\mathbb{D}.\) It is immediate from (3.13) that \(f\) is locally univalent in \(\mathbb{D}.\) Let \(\omega \in \mathcal{B}(\Lambda),\) where \(\lambda \in (0,\cos \delta],\) and let \(g\) be the solution of the differential equation
\[
-(1 - z)^2 g'(z) = 4 \frac{1 - \omega(z)}{1 + \omega(z)} g(z), \quad z \in \mathbb{D},
\]
with the boundary condition \(g_z(1) = 0.\) As was proved in [7, Theorem 3], \(g\) belongs to the class \(\mathcal{F}_0^*,\) so it is univalent, and \(g(\mathbb{D})\) being a simply connected domain lies in a
wedge of angle $2\lambda\pi$. Hence there exists a single-valued analytic branch of $\log g$ in $\mathbb{D}$, and

$$g^{e^{-i\delta}}(z) = \exp \{e^{-i\delta} \log g(z)\}, \quad z \in \mathbb{D},$$  \hspace{1cm} (3.35)

is well defined. But, in view of (3.13) and (3.34), we have

$$\frac{g'}{g} = e^{i\delta} \frac{f'}{f},$$  \hspace{1cm} (3.36)

so

$$f = g^{e^{-i\delta}}.$$  \hspace{1cm} (3.37)

Since $\lambda \in (0, \cos \delta]$, from the above, the univalence of $f$ in $\mathbb{D}$ follows.

Now, we prove that $f(\mathbb{D}) \in \mathcal{F}_{0}^{*}(\delta)$. This is clear, looking at the relation (3.37) between classes $\mathcal{F}_{0}^{*}$ and $\mathcal{F}_{0}^{*}(\delta)$, which yields the geometric relation between starlikeness and spirallikeness of domains in the plane. To be self-contained, we prove it without using geometric properties of functions in $\mathcal{F}_{0}^{*}$. We assume that $\delta \neq 0$, since this case reduces to [7, Theorem 3].

Let $O_k = \partial \mathbb{D} \setminus \{1\}$ and $\Gamma_k = \partial f(\mathbb{D})$ for every $k > 0$. Suppose, on the contrary, that $f(\mathbb{D}) \notin \mathcal{F}_{0}^{*}(\delta)$. By Theorem 3.5, there exists $k > 0$ such that $f(\mathbb{D}) \notin \mathcal{F}_{0}^{*}(\delta)$. Hence $w_0 L(\delta) \notin f(\mathbb{D})$ for some $w_0 \in f(\mathbb{D})$. Thus there exists $w_1 \in (w_0 L(\delta) \setminus \{w_0\}) \cap \Gamma_k$ such that the subarc of $w_0 L(\delta)$ joining $w_1$ and $w_0$ without $w_1$ is contained in $f(\mathbb{D})$. Since $w_1 \in \Gamma_k$, $w_1 = f(z_1)$ for some $z_1 \in O_k$. Let

$$v(t) = w_0 \exp \{e^{-i\delta} t\}, \quad t \leq 0,$$  \hspace{1cm} (3.38)

be a parametrization of $w_0 L(\delta)$. Clearly,

$$w_1 = v(t_1) = w_0 \exp \{e^{-i\delta} t_1\}$$  \hspace{1cm} (3.39)

for some $t_1 < 0$. Let

$$w(t) = w_1 \exp \{e^{-i\delta} s\}, \quad s \leq 0,$$  \hspace{1cm} (3.40)

be a parametrization of $w_1 L(\delta)$. From (3.38), (3.39), and (3.40), we have

$$w(t) = w_0 \exp \{e^{-i\delta} t_1\} \exp \{e^{-i\delta} s\} = w_0 \exp \{e^{-i\delta} (t_1 + s)\},$$  \hspace{1cm} (3.41)

which means that $w_1 L(\delta)$ is a subset of $w_0 L(\delta)$. Moreover,

$$v'(t_1) = e^{-i\delta} w_0 \exp \{e^{-i\delta} t_1\} = e^{-i\delta} f(z_1) = w'(0).$$  \hspace{1cm} (3.42)

Therefore the tangent line $l$ to $w_0 L(\delta)$ at $w_1$ has the directional vector $v'(t_1) = w'(0)$ and is the boundary of two closed half-planes denoted by $H_1$ and $H_2$. One of them, say $H_1$, contains the origin. Let $\delta \in (-\pi/2, 0)$. As we remarked in the proof of Theorem 3.6,
the spiral $L(\delta)$ has the shape such that it attains 1 from the lower half-plane. Moreover, $w_1L(\delta)$ parametrized as above turns round the origin in the counterclockwise direction. Hence, $iw'(0)$ lies in $H_1$. Observe that either $\Gamma_k$ is tangent as well to $w_1L(\delta)$ (one-sided) as to $l$ at $w_1$ and then $\tau(z_1)$ lies in $l$, or, by [9, Proposition 2.13, page 28], there is a crosscut $C \subset l$ of $f(\Omega_k)$ with one endpoint at $w_1$. Thus, by [9, Proposition 2.12, page 27], $f(\Omega_k)$ has exactly two components, one of them, say $G$, lies in $H_2$. Moreover, $\partial G = C \cup \Gamma$, where $\Gamma \subset \Gamma_k$ ends at $w_1$. Hence $\Gamma$ is a subset of $H_2$ and, since it is part of a positively oriented closed analytic curve $\Gamma_k$, we deduce finally that the tangent vector $\tau(z_1)$ to $\Gamma_k$ at $f(z_1)$ lies in $H_2$. Since $iw'(0)$ is orthogonal to $l$ and lies in $H_1$, we deduce that

$$|\varphi(z_1)| \geq \frac{\pi}{2},$$

(3.43)

where $\varphi(z_1)$ denotes the directed angle defined by (3.19), with $z_1$ instead of $z$. This contradicts (3.13). Similarly, we get a contradiction assuming that $\delta \in (0, \pi/2)$. 

**Remark 3.9.** In [1], the authors found necessary and sufficient conditions for functions to be in $F_0^*(\delta)$ (Theorem 2.1). The analytic formula (2.1) in [1] generalizes the Robertson inequality for starlike functions with respect to a boundary point. In fact, the authors of [1] proved that each spirallike function with respect to a boundary point is a complex power of a corresponding function which is starlike with respect to a boundary point. Formula (3.13) presents an alternative analytic description of the class $F_0^*(\delta)$. In case $\delta = 0$ ($\mu = 2\pi$ in [1, equation (2.1)]), these two analytic formulas for $F_0^*(\delta)$ characterizing starlike functions with respect to a boundary point are equivalent. Looking at [1, Theorem 2.1(III) and Theorems 3.2 and 3.3], we can expect that formulas (2.1) in [1] and (3.13) of the present paper are equivalent, which, in fact, means that in Theorem 3.6 the assumptions $\lambda \in (0, 1]$ should be replaced by $\lambda \in (0, \cos \delta']$. This is an open problem.

3.4. Now, we present some examples of functions. In all of the examples below, $\delta \in (-\pi/2, \pi/2)$ and $p(z) = -e^{i\delta}(1-z)^2f'(z)/f(z)$. It is convenient to express formula (3.13) in terms of the class $\mathcal{P}(\lambda)$. Therefore, in the examples below, we apply Remark 2.6 which says that $p \in \mathcal{P}(\lambda)$ if and only if $p_\perp(1) = 0$ and $p_\perp'(1) = -2\lambda$. In every case, we use Theorem 3.8 reformulated by using the class $\mathcal{P}(\lambda)$.

**Example 3.10.** (1) $f(z) = (1-z)/(1+z)$, $\beta > 0$, $z \in \mathbb{D}$.

Then $p(z) = 2\beta(1-z)/(1+z)$. Hence $\Re p(z) > 0$, $z \in \mathbb{D}$, $p(1) = 0$, and $p'(1) = -\beta$. Consequently, $f \in F_0^*(\delta)$ for $\beta \in (0, 2]$. For every $\beta > 2$, $f \notin F_0^*(\delta)$.

(2) $f(z) = (1-z)e^{-i\beta}$, $\beta > 0$, $z \in \mathbb{D}$.

Then $p(z) = \beta(1-z)$. Hence $\Re p(z) > 0$, $z \in \mathbb{D}$, $p(1) = 0$, and $p'(1) = -\beta$. Consequently, $f \in F_0^*(\delta)$ for $\beta \in (0, 2]$. For every $\beta > 2$, $f \notin F_0^*(\delta)$.

(3) $f(z) = (1-z)^2e^{-i\beta}e^{-i\beta}z$, $z \in \mathbb{D}$.

Then $p(z) = -z^2 + 1$. Hence $\Re p(z) > 0$, $z \in \mathbb{D}$, $p(1) = 0$, and $p'(1) = -2$. Consequently, $f \in F_0^*(\delta)$.

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