Concerning the Goldberg conjecture, we will prove a result obtained by applying the result of Iton in terms of $L^2$-norm of the scalar curvature.

2000 Mathematics Subject Classification: 53C25, 53C55.

1. Introduction. An almost Hermitian manifold $M$ is called an almost Kähler manifold if the corresponding Kähler form is a closed 2-form. It is well known that an almost Kähler manifold with integrable almost-complex structure is Kählerian. Concerning the integrability of almost Kähler manifold, the following conjecture by Goldberg is known (see [2]).

**Conjecture 1.1.** A compact almost Kähler-Einstein manifold is Kählerian.

Sekigawa [8] proved that the conjecture is true if the scalar curvature $\tau$ of $M$ is nonnegative. But the conjecture is still open in the case where $\tau$ is negative. Recently, applying the Seiberg-Witten theory, Itoh [4] obtained the following integrability condition for certain almost Kähler-Einstein 4-manifolds in terms of the $L^2$-norm of the scalar curvature.

**Theorem 1.2** [4]. Let $M$ be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If $M$ satisfies

$$\int_M \tau^2 dV = 32\pi^2 (2\chi(M) + p_1(M)),$$

then it must be Kähler-Einstein. Here, $\chi(M)$ and $p_1(M)$ are the Euler characteristic and the first Pontrjagin number of $M$, respectively.

As a corollary, he also proved the following.

**Corollary 1.3** [4]. Let $M$ be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If $M$ satisfies

$$\int_M \tau^2 dV \leq 24 \int_M \|W^+\|^2 dV,$$

or, more strictly, if $|\tau| \leq 2\sqrt{6}\|W^+\|$ at each point of $M$, then $M$ must be Kähler-Einstein. Here, $W^+$ is the self-dual Weyl curvature operator of the metric $g$. 
In this paper, concerning the Goldberg conjecture, we will prove a result obtained by using Corollary 1.3 (see Theorem 2.2).

2. Preliminaries and the result. Let \( M = (M, J, g) \) be a four-dimensional almost Kähler-Einstein manifold with the almost-complex structure \( J \) and the Hermitian metric \( g \). We denote by \( \Omega \) the Kähler form of \( M \) defined by \( \Omega(X, Y) = g(X, JY) \) for \( X, Y \in \mathfrak{X}(M) \), the set of all smooth vector fields on \( M \). We assume that \( M \) is oriented by the volume form \( dV = \Omega^2/2 \). We denote by \( \nabla, R, \rho, \) and \( \tau \) the Riemannian connection, the curvature tensor, the Ricci tensor, and the scalar curvature of \( M \), respectively. We assume that the curvature tensor is defined by \( R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \) for \( X, Y, Z \in \mathfrak{X}(M) \). We denote by \( \rho^* \) the Ricci \(*\)-tensor of \( M \) defined by

\[
\rho^*(x, y) = \frac{1}{2} \text{ trace of } (z \mapsto R(x, Jy)Jz)
\] (2.1)

for \( x, y, z \in T_pM \), the tangent space of \( M \) at \( p \in M \). The Ricci \(*\)-tensor satisfies \( \rho^*(x, y) = \rho^*(Jy, Jx) \) for any \( x, y \in T_pM, p \in M \). We note that if \( M \) is Kählerian, the Ricci tensor and the Ricci \(*\)-tensor coincide on \( M \). The \(*\)-scalar curvature \( \tau^* \) of \( M \) is the trace of the linear endomorphism \( Q^* \) defined by \( g(Q^*x, y) = \rho^*(x, y) \) for \( x, y \in T_pM, p \in M \). Since \( \|\nabla J\|^2 = 2(\tau^* - \tau) \), \( M \) is a Kähler manifold if and only if \( \tau^* = \tau = 0 \) on \( M \). An almost Hermitian manifold \( M \) is called a weakly \(*\)-Einstein manifold if \( \rho^* = \lambda^* g \) (\( \lambda^* = \tau^*/4 \)) and a \(*\)-Einstein if \( M \) is weakly \(*\)-Einstein with constant \(*\)-scalar curvature. The following identity holds for any four-dimensional almost Hermitian Einstein manifold:

\[
\frac{1}{2} \{\rho^*(x, y) + \rho^*(y, x)\} = \frac{\tau^*}{4} g(x, y)
\] (2.2)

for \( x, y \in T_pM, p \in M \).

Now, let \( \wedge^2 M \) be the vector bundle of all real 2-forms on \( M \). The bundle \( \wedge^2 M \) inherits a natural inner product \( g \) and we have an orthogonal decomposition

\[
\wedge^2 M = \mathbb{R} \Omega \oplus LM \oplus \wedge^{1,1}_0 M,
\] (2.3)

where \( LM \) (resp., \( \wedge^{1,1}_0 M \)) is the bundle of \( J \)-skew-invariant (resp., \( J \)-invariant) 2-forms on \( M \) perpendicular to \( \Omega \). We can identify the subbundle \( \mathbb{R} \Omega \oplus LM \) (resp., \( \wedge^{1,1}_0 M \)) with the bundle \( \wedge^2 M \) (resp., \( \wedge^2 M \)) of self-dual (resp., anti-self-dual) 2-forms on \( M \). Since \( M \) is Einstein, it is well known that the curvature operator \( \mathcal{R} : \wedge^2 M \to \wedge^2 M \) preserves \( \wedge^2 M \) and that the Weyl curvature operator \( \mathcal{W} : \wedge^2 M \to \wedge^2 M \) is given by

\[
\mathcal{W}(\iota(X) \wedge \iota(Y)) = \mathcal{R}(\iota(X) \wedge \iota(Y)) - \frac{\tau}{12} \iota(X) \wedge \iota(Y),
\] (2.4)

where \( \iota \) is the duality between the tangent bundle and the cotangent bundle of \( M \) by means of the metric \( g \). Let \( \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\} \) be a (local) unitary frame field and put \( e^i = \iota(e_i) \). Then, the Kähler form is represented by \( \Omega = -e^1 \wedge e^2 - e^3 \wedge e^4 \). Further,
we see that
\[
\{\Phi, J\Phi\} = \left\{ \frac{1}{\sqrt{2}} (e^1 \wedge e^3 - e^2 \wedge e^4), \frac{1}{\sqrt{2}} (e^1 \wedge e^4 + e^2 \wedge e^3) \right\},
\]
\[
\{\Psi_1, \Psi_2, \Psi_3\} = \left\{ \frac{1}{\sqrt{2}} (e^1 \wedge e^2 - e^3 \wedge e^4), \frac{1}{\sqrt{2}} (e^1 \wedge e^3 + e^2 \wedge e^4), \frac{1}{\sqrt{2}} (e^1 \wedge e^4 - e^2 \wedge e^3) \right\}
\]
are (local) orthonormal bases of $LM$ and $\wedge_0^{1,1} M = \wedge_1^2 M$, respectively.

In this paper, for any orthonormal basis (resp., any local orthonormal frame field) \{e_1, e_2, e_3, e_4\} of a point $p \in M$ (resp., on a neighborhood of $p$), we will adopt the following notational convention:

\[
J_{ij} = g(J_{e_i} e_j), \quad \Gamma_{ij} = g(\nabla_{e_i} e_j, e_k),
\]
\[
R_{ijkl} = g(R(e_i, e_j) e_k, e_l), ..., R_{ijkl} = g(R(J_{e_i} e_j) e_k, e_l),
\]
\[
\rho_{ij} = \rho(e_i, e_j), ..., \rho_{ij} = \rho(e_i, e_j), \quad \rho^*_{ij} = \rho^*(e_i, e_j),
\]
\[
\nabla_i J_{jk} = g((\nabla_{e_i} J) e_j, e_k), ..., \nabla_i J_{jk} = g((\nabla_{e_i} J) e_j, e_k),
\]
and so on, where the Latin indices run over the range 1, 2, 3, 4. We define functions $A$, $B$, $C$, $D$, $G$, and $K$ on $M$ by

\[
A = \sum_{i,j,k,l,a=1}^4 (\nabla_a J_{ij}) R_{ijkl}(\nabla_a J_{kl}),
\]
\[
B = \sum_{i,j,k,l,a=1}^4 (\nabla_a J_{ij})(\nabla_a J_{kl})(\nabla_b J_{ij})(\nabla_b J_{kl}),
\]
\[
C = \sum_{i,j,k,l=1}^4 R_{ijkl} R_{ijkl}, \quad D = \sum_{i,j,k,l=1}^4 (R_{ijkl} - R_{ijlk})^2,
\]
\[
G = \sum_{i,j=1}^4 (\rho^*_{ij} - \rho_{ij})^2, \quad K = (u - v)^2 + 4w^2,
\]

where $u = g(\bar{\mathcal{R}}(\Phi), \Phi)$, $v = g(\bar{\mathcal{R}}(J\Phi), J\Phi)$, and $w = g(\bar{\mathcal{R}}(\Phi), J\Phi)$. First, we will prove the following.

**Lemma 2.1.** The norm of the self-dual Weyl operator $W^+$ is given by

\[
||W^+||^2 = \frac{1}{16} \left( G + D + (\tau^*)^2 - \frac{\tau^2}{3} \right).
\]

**Proof.** Let \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\} be any (local) unitary frame field on $M$ and we put $\Omega_0 = -\Omega/\sqrt{2} = (e^1 \wedge e^2 + e^3 \wedge e^4)/\sqrt{2}$, $\Phi = (e^1 \wedge e^3 - e^2 \wedge e^4)/\sqrt{2}$, and $J\Phi = (e^1 \wedge e^4 + e^2 \wedge e^3)/\sqrt{2}$. Then, \{\Omega_0, \Phi, J\Phi\} is an orthonormal basis of $\wedge_1^2 M$. Thus, we have

\[
||W^+||^2 = g(W^+(\Omega_0), \Omega_0)^2 + g(W^+(\Omega_0), \Phi)^2 + g(W^+(\Omega_0), J\Phi)^2
\]
\[+ g(W^+(\Phi), \Omega_0)^2 + g(W^+(\Phi), \Phi)^2 + g(W^+(\Phi), J\Phi)^2
\]
\[+ g(W^+(J\Phi), \Omega_0)^2 + g(W^+(J\Phi), \Phi)^2 + g(W^+(J\Phi), J\Phi)^2.
\]
Taking account of (2.4), we have

\[
g(W^+(\Omega_0), \Omega_0) = \frac{1}{2} \left( -R_{1212} - 2R_{1243} - R_{3434} - \frac{\tau}{6} \right) = \frac{1}{12} (3\tau^* - \tau),
\]
\[
g(W^+(\Omega_0), \Phi) = \frac{1}{2} ( -R_{1213} - R_{1224} - R_{3413} - R_{3424} ) = -\frac{1}{2} (\rho_{14}^* - \rho_{41}^*),
\]
\[
g(W^+(\Omega_0), J\Phi) = \frac{1}{2} ( -R_{1214} - R_{1223} - R_{3414} - R_{3423} ) = \frac{1}{2} (\rho_{13}^* - \rho_{31}^*),
\]
\[
g(W^+(\Phi), \Phi) = \frac{1}{2} ( -R_{1313} + 2R_{1324} - R_{2424} - \frac{\tau}{6} ) = -(R_{1313} - R_{1324}) - \frac{\tau}{12},
\]
\[
g(W^+(\Phi), J\Phi) = \frac{1}{2} ( -R_{1314} - R_{1323} + R_{2414} + R_{2423} ) = -(R_{1314} + R_{1323}),
\]
\[
g(W^+(J\Phi), J\Phi) = \frac{1}{2} ( -R_{1414} - 2R_{1423} - R_{2323} - \frac{\tau}{6} ) = -(R_{1414} + R_{1423}) - \frac{\tau}{12}.
\]

Thus, we have

\[
||W^+||^2 = \frac{1}{12^2} (3\tau^* - \tau)^2 + \frac{2\tau^2}{12^2} + \frac{1}{2} (\rho_{13}^* - \rho_{31}^*)^2 + \frac{1}{2} (\rho_{14}^* - \rho_{41}^*)^2
\]
\[
+ (R_{1313} - R_{1324})^2 + (R_{1314} + R_{1323})^2 + (R_{1314} + R_{1323})^2
\]
\[
+ (R_{1414} + R_{1423})^2 + \frac{\tau}{6} (R_{1313} - R_{1324} + R_{1414} + R_{1423})
\]
\[
= \frac{1}{12^2} (3\tau^* - \tau)^2 + \frac{2\tau^2}{12^2} + \frac{G}{8}
\]
\[
+ \frac{1}{4} \sum_{i<j<k<l} (R_{ijkl} - R_{ijkl})^2 - \frac{1}{4} \sum_{k<l} (R_{12kl} - R_{12kl})^2
\]
\[
- \frac{1}{4} \sum_{k<l} (R_{34kl} - R_{34kl})^2 - \frac{\tau}{6} \left( -\frac{\tau}{4} - R_{1212} - R_{1234} \right)
\]
\[
= \frac{1}{12^2} (3\tau^* - \tau)^2 + \frac{2\tau^2}{12^2} + \frac{G}{8} + \frac{D}{16} - \frac{G}{32} - \frac{G}{32} + \frac{\tau}{6} \left( -\frac{\tau}{4} + \frac{\tau^*}{4} \right)
\]
\[
= \frac{D}{16} + \frac{G}{16} + \frac{(\tau^*)^2}{16} - \frac{\tau^2}{48}.
\]

The lemma follows. \(\square\)

Next, we recall the following equalities established in [6]:

\[
A = \frac{1}{4} B = \frac{(\tau^* - \tau)^2}{2},
\]
\[
C = -2K + \frac{(\tau^* - \tau)^2}{8},
\]
\[
G = 4||\rho^*||^2 - (\tau^*)^2 = 16 \left\{ (\rho_{13}^*)^2 + (\rho_{14}^*)^2 \right\},
\]
\[
K = (u + v)^2 + 4(w^2 - uv) = \frac{(\tau^* - \tau)^2}{16} - 4 \det R'_{LM},
\]
\[
||R'_{LM}||^2 = \frac{1}{16} D,
\]
\[
||R'_{LM}||^2 = \frac{1}{16} (D - G),
\]
\[
(2.12)
\]
where \( \mathcal{R}_{LM} \) is the restriction of \( \mathcal{R} \) to \( LM \) and \( \mathcal{R}'_{LM} = P_{LM} \circ \mathcal{R}_{LM} \), the composition of \( \mathcal{R}_{LM} \) and the natural projection \( P_{LM} : \wedge^2 M \to LM \). We define a vector field \( \eta = (\eta_a) \) on \( M \) by \( \eta_a = \sum_{i,j=1}^{4} (\nabla a J_{ij}) \rho^i \bar{j} \), then we obtain the following (see [6, (2.23)]):

\[
\Delta \tau^* = \frac{G}{2} + 4K + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} - 4 \operatorname{div} \eta. \tag{2.13}
\]

Further, from (2.12) and the curvature identity

\[
R_{ijkl} - R_{ijkl} - R_{ikjl} + R_{ikjl} - R_{ijkl} + R_{ijkl} + R_{ijkl} = 2 \sum_{a=1}^{4} (\nabla a J_{ij}) \nabla a J_{kl} \tag{2.14}
\]

by Gray [3] for almost Kähler manifold, we have

\[
A = \frac{1}{2} \sum R_{ijkl}(R_{ijkl} - R_{ijkl} - R_{ijkl} + R_{ikjl} + R_{ikjl} + R_{ijkl} + R_{ijkl})
\]

\[
= \frac{1}{4} \sum (R_{ijkl} - R_{ijkl})^2 - \frac{1}{2} \sum (R_{ijkl} - R_{ijkl})(R_{ijkl} - R_{ijkl}) + 2 \sum R_{ijkl} R_{ijkl}
\]

\[
= \frac{D}{4} - \frac{1}{4} \left\{ -16||\mathcal{R}'_{LM}||^2 + \sum (R_{ij12} + R_{ij34} - R_{ij12} - R_{ij34})^2 \right\} + 2C \tag{2.15}
\]

Thus, from (2.12) and this equality, we obtain

\[
\frac{D}{2} - \frac{G}{2} - 4K + \frac{(\tau^* - \tau)^2}{4} = 0. \tag{2.16}
\]

Now, we are ready to prove the following.

**Theorem 2.2.** Let \( M = (M, J, g) \) be a four-dimensional compact almost Kähler-Einstein manifold with negative scalar curvature. If \( M \) satisfies

\[
\int_M \{G + \tau (\tau^* - \tau)\} dV \geq 0, \tag{2.17}
\]

or, more strictly, if \( \tau^* - \tau \leq -G/\tau \) at each point of \( M \), then \( M \) is Kähler-Einstein.

**Proof.** From (2.8), we have

\[
24 \int_M ||\mathcal{W}||^2 dV - \int_M \tau^2 dV = \frac{3}{2} \int_M \{G + D + (\tau^* - \tau)(\tau^* + \tau)\} dV. \tag{2.18}
\]

On one hand, from (2.13) and (2.16), we have

\[
0 = \int_M \left\{ \frac{G}{2} + 4K + \frac{(3\tau^* - \tau)(\tau^* - \tau)}{4} \right\} dV = \int_M \left\{ \frac{D}{2} + \frac{\tau^* (\tau^* - \tau)}{2} \right\} dV. \tag{2.19}
\]
Thus, from (2.18) and (2.19), we obtain
\[
24 \int_M \| W^+ \|^2 dV - \int_M \tau^2 dV = \frac{2}{3} \int_M \{ G + \tau (\tau^* - \tau) \} dV.
\]  
(2.20)
Therefore, from Corollary 1.3, the assertion of the theorem immediately follows.

**Remark 2.3.** The above theorem is concerned with the following facts.

1. For a compact four-dimensional almost Kähler-Einstein manifold, the function \( \tau^* - \tau \) vanishes at some point of \( M \) (see [1, 5]).
2. A four-dimensional compact almost Kähler-Einstein and weakly \(*\)-Einstein manifold \((G \equiv 0)\) is a Kähler manifold (see [7]).
3. Let \( M \) be a four-dimensional compact strictly almost Kähler-Einstein, but not weakly \(*\)-Einstein manifold. Then, we see that \( G > 0 \) on \( M_0 = \{ p \in M \mid \tau^* - \tau > 0 \} \), and hence \( \tau^* - \tau = 0 \) at which \( G = 0 \) (see [5]).

**References**


R. S. Lemence: Department of Mathematical Science, Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan

E-mail address: f02n406n@mail.cc.niigata-u.ac.jp

T. Oguro: Department of Mathematical Sciences, School of Science and Engineering, Tokyo Denki University, Saitama 350-0394, Japan

E-mail address: oguro@r.dendai.ac.jp

K. Sekigawa: Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan

E-mail address: sekigawa@sc.niigata-u.ac.jp