ON FURTHER STRENGTHENED HARDY-HILBERT’S INEQUALITY

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We obtain an inequality for the weight coefficient \( \omega(q,n) \) \( (q > 1, 1/p + 1/q = 1, n \in \mathbb{N}) \) in the form
\[
\omega(q,n) = \frac{1}{\pi} \sin\left(\frac{\pi}{p}\right) - \frac{1}{n^{1/p} + (2/a)n^{-1/q}}
\]
where \( 0 < a < 147/45 \), as \( n \geq 3 \); \( 0 < a < (1 - C)/(2C - 1) \), as \( n = 1, 2 \), and \( C \) is an Euler constant. We show a generalization and improvement of Hilbert’s inequalities. The results of the paper by Yang and Debnath are improved.

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1. Introduction. The following inequalities are well known as Hardy-Hilbert’s inequalities:
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left( \sum_{n=1}^{\infty} a_n^p \right)^{1/p} \left( \sum_{n=1}^{\infty} b_n^q \right)^{1/q},
\]
(1.1)
\[
\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^p < \left( \frac{\pi}{\sin(\pi/p)} \right)^p \sum_{n=1}^{\infty} a_n^p.
\]
(1.2)
In recent years, Gao [1, 2], Xu and Guo [4], Hsu and Wang [3], Yang [6], and Yang and Gao [7] gave some distinct improvements of (1.1). Yang and Debnath [5] gave a strengthened version by the following inequality:

\[
\omega(q,n) = \sum_{m=1}^{\infty} \frac{1}{m+n} \left( \frac{n}{m} \right)^{1/q} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}}.
\]
(1.3)
In this paper, we show a new generalization and improvement of (1.1) by improving (1.3).

First we introduce some lemmas.

**Lemma 1.1** [2]. Let \( f(x) > 0, f^{(2r-1)}(x) < 0, f^{(2r)}(x) \geq 0, x \in [1, \infty), r = 1, 2, f^{(r)}(\infty) = 0, r = 0, 1, 2, 3, 4, \) and \( \int_1^{\infty} f(x)dx < \infty, \) then
\[
\sum_{m=1}^{\infty} f(m) \leq \int_1^{\infty} f(x)dx + \frac{1}{2} f(1) - \frac{1}{12} f''(1).
\]
(1.4)

**Lemma 1.2** [5]. Let \( q > 1, 1/p + 1/q = 1, n \in \mathbb{N}, \) then
\[
\omega(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}} [f_n(p) + g_n(p)],
\]
(1.5)
where $\omega(q,n)$ is defined by (1.3), and

$$f_n(p) := p + \frac{1}{12p} + \frac{1}{(1+p)n} + \frac{1}{12pn^2} + \frac{1}{3(1+3p)n^3},$$

$$g_n(p) := -\frac{1}{12pn} - \frac{1}{2(1+2p)n^2} - \frac{7}{12} - \frac{1}{2n} + \frac{1}{12n^2} - \frac{7}{12n^3}. \quad (1.6)$$

**Lemma 1.3** [5]. Let $p > 1$, $n \in \mathbb{N}$, then

$$f_n(p) + g_n(p) > \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}. \quad (1.7)$$

**Lemma 1.4**. Let $q > 1$, $1/p + 1/q = 1$, $n \in \mathbb{N}$, then

$$\omega(q,n) =: \sum_{m=1}^{\infty} \frac{1}{m+n} \left( \frac{n}{m} \right)^{1/q} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/q}},$$

$$\omega(p,n) =: \sum_{m=1}^{\infty} \frac{1}{m+n} \left( \frac{n}{m} \right)^{1/p} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/p}}, \quad (1.8)$$

where $0 < a < 147/45$ as $n \geq 3$; $0 < a < (1-C)/(2C-1)$ as $n = 1, 2$, and $C$ is an Euler constant.

**Proof.** For $n \geq 3$,

$$\left( \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3} \right) \left( x + \frac{1}{ny} \right) = \frac{x}{2} + \frac{1}{n} \left( \frac{1}{2y} - \frac{x}{12} - \frac{x}{12yn} - \frac{x}{2n^2} - \frac{1}{2yn^3} \right), \quad (1.10)$$

where $x > 0$, $y > 0$, $xy = a$.

We first prove

$$\frac{1}{2y} - \frac{x}{12} - \frac{1}{12yn} - \frac{x}{2n^2} - \frac{1}{2yn^3} = \frac{6n^2 - xyn^3 - n^2 - 6xyn - 6}{12yn^3} > 0. \quad (1.11)$$

Formula (1.11) is equivalent to $\psi(n) = 6n^3 - xyn^3 - n^2 - 6xyn - 6 > 0$.

Since $\psi'(x) = 18x^2 - 3ax^2 - 2x - 6a$, $\psi''(x) = 36x - 6ax - 2$. When $0 < a < 147/45$, $\psi''(x) = 36x - 6ax - 2 > 0$, $\psi'(3) = 156 - 33a > 0$, then $\psi'(x) > 0$ and $\psi(3) = 147 - 45a > 0$, hence $\psi(n) > 0$ for $n \geq 3$. Therefore $(1/2 - 1/12n - 1/2n^3)(x + 1/ny) > x/2$. Namely, $1/2 - 1/12n - 1/2n^3 > 1/(2 + 2(an)^{-1})$, for $n \geq 3$. By (1.5) and (1.7), we have

$$\omega(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}} \left( \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3} \right),$$

$$< \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/q}}. \quad (1.12)$$
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Since $\phi(x) = 1/(2x + 2(xn)^{-1})$ is strictly increasing on $(0, \infty)$ and $0 < a < (1 - C)/(2C - 1)$, by $\omega(q, n) < \pi/\sin(\pi/p) - (1 - C)/n^{1/p}$ (see [7]), we have, when $n = 1$,

$$\omega(q, 1) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{1} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2 + 2(2C - 1)/(1 - C)}$$

(1.13)

when $n = 2$,

$$\omega(q, 2) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2^{1/p}} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2^{1/p} 2 + (2/a)(1/2)}.$$  

(1.14)

By (1.12), (1.13), and (1.14), (1.8) is valid for any $n \in \mathbb{N}$. Interchanging $p, q$ in (1.8), since $\pi/\sin(\pi/p) = \pi/\sin(\pi/q)$, we have (1.9). The lemma is proved. \qed

2. Main results. Now we introduce main results.

**Theorem 2.1.** Let $p > 1$, $1/p + 1/q = 1$, $a_n \geq 0$, $b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. Then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left( \left( \frac{\pi}{\sin(\pi/p)} - \frac{b}{2 + 2b} \right) a_1^p + \left( \frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/p}(2b+2)} \right) a_2^p \right. \left. + \sum_{n=3}^{\infty} \left( \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{1/q}} \right) a_n^p \right)^{1/p} \times \left( \left( \frac{\pi}{\sin(\pi/p)} - \frac{b}{2 + 2b} \right) b_1^q + \left( \frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/q}(2b+2)} \right) b_2^q \right. \left. + \sum_{n=3}^{\infty} \left( \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + (2/a)n^{1/p}} \right) b_n^q \right)^{1/q},$$

(2.1)

\[
\sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{a_n}{m+n} \right)^p < \left( \frac{\pi}{\sin(\pi/p)} \right)^{p-1} \left( \left( \frac{\pi}{\sin(\pi/p)} - \frac{b}{2 + 2b} \right) a_1^p + \left( \frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/p}(2b+2)} \right) a_2^p \right. \left. + \sum_{n=3}^{\infty} \left( \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{1/q}} \right) a_n^p \right),
\]

(2.2)

where $0 < a < 147/45$, $0 < b < (1 - C)/(2C - 1)$. 


In particular, when \( a = b = e \), \( e \) is a constant,

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left( \sum_{n=1}^{\infty} \left( \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/e)n^{-1/q}} \right) a_n^p \right)^{1/p} \\
\times \left( \sum_{n=1}^{\infty} \left( \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + (2/e)n^{-1/p}} \right) b_n^q \right)^{1/q},
\]

(2.3)

By Hölder’s inequality, we have

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left( \sum_{n=1}^{\infty} \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/e)n^{-1/q}} \right) a_n^p.
\]

(2.4)

By (1.8) and (1.9), (2.1) is valid.

By Hölder’s inequality and (1.9), we have

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m}{(m+n)^{1/p}} \left( \frac{m}{n} \right)^{1/pq} \frac{b_n}{(m+n)^{1/q}} \left( \frac{n}{m} \right)^{1/pq}
\]

\[
\leq \left( \sum_{n=1}^{\infty} \frac{a_n^{pq}}{m+n} \left( \frac{m}{n} \right)^{1/q} \right)^{1/p} \left( \sum_{n=1}^{\infty} \frac{b_n^{pq}}{m+n} \left( \frac{n}{m} \right)^{1/pq} \right)^{1/q}
\]

(2.5)

\[
= \left( \sum_{n=1}^{\infty} \omega(q,n) a_n^p \right)^{1/p} \left( \sum_{n=1}^{\infty} \omega(p,n) b_n^q \right)^{1/q}.
\]

Then

\[
\sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{a_n}{m+n} \right)^{p < \left( \frac{\pi}{\sin(\pi/p)} \right)^{p/q} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n^{pq}}{m+n} \left( \frac{n}{m} \right)^{1/q}
\]

(2.6)

By (1.8), (2.2) is valid. The theorem is proved.

\[\square\]

**Remark 2.2.** As \( a = b = 0 \), inequalities (2.1) and (2.2) change to (1.1) and (1.2), respectively, hence inequalities (2.1) and (2.2) are generalization and improvement of (1.1) and (1.2), respectively.
Remark 2.3. As $a = b = 2$, inequalities (2.1) and (2.2) change to (1.11) and (3.1) in [5], respectively, hence inequalities (2.1) and (2.2) are generalization and improvement of (1.11) and (3.1) in [5], respectively.

Remark 2.4. We give an open question: how to determine the constant $a$ such that $1/(2n^{1/p} + (2/a)n^{-1/q})$ is best possible.

References


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