ON UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN OPERATOR

F. M. AL-OBOUDI

Received 17 August 2001 and in revised form 18 March 2002

We introduce a class of univalent functions \( R^n(\lambda, \alpha) \) defined by a new differential operator \( D^n f(z) \), \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \), where \( D^0 f(z) = f(z) \), \( D^1 f(z) = (1 - \lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \), \( \lambda \geq 0 \), and \( D^n f(z) = D_\lambda (D^{n-1} f(z)) \). Inclusion relations, extreme points of \( R^n(\lambda, \alpha) \), some convolution properties of functions belonging to \( R^n(\lambda, \alpha) \), and other results are given.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let \( A \) denote the class of functions of the form

\[
  f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)
\]

analytic in the unit disc \( \Delta = \{ z : |z| < 1 \} \).

We denote by \( R(\alpha) \) the subclass of \( A \) for which \( \Re f'(z) > \alpha \) in \( \Delta \). For a function \( f \in A \), we define the following differential operator:

\[
  D^0 f(z) = f(z), \\
  D^1 f(z) = (1 - \lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \geq 0, \\
  D^n f(z) = D_\lambda (D^{n-1} f(z)). \quad (1.4)
\]

If \( f \) is given by (1.1), then from (1.3) and (1.4) we see that

\[
  D^n f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + (k - 1)\lambda \right]^n a_k z^k. \quad (1.5)
\]

When \( \lambda = 1 \), we get Sălăgean’s differential operator [8].

Let \( R^n(\lambda, \alpha) \) denote the class of functions \( f \in A \) which satisfy the condition

\[
  \Re (D^n f(z))^\top > \alpha, \quad z \in \Delta, \quad (1.6)
\]

for some \( 0 \leq \alpha \leq 1 \), \( \lambda \geq 0 \), and \( n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). It is clear that \( R^0(\lambda, \alpha) \equiv R(\alpha) \equiv R^n(0, \alpha) \) and that \( R^1(\lambda, \alpha) \equiv R(\lambda, \alpha) \), the class of functions \( f \in A \) satisfying

\[
  \Re (f''(z) + \lambda z f'''(z)) > \alpha, \quad z \in \Delta, \quad (1.7)
\]

studied by Ponnusamy [5] and others.
The Hadamard product or convolution of two power series \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( g(z) = \sum_{k=0}^{\infty} b_k z^k \) is defined as the power series \( (f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k \), \( z \in \Delta \).

The object of this paper is to derive several interesting properties of the class \( R^n(\lambda, \alpha) \) such as inclusion relations, extreme points, some convolution properties, and other results.

2. Inclusion relations. Theorem 2.3 shows that the functions in \( R^n(\lambda, \alpha) \) belong to \( R(\alpha) \) and hence are univalent. We need the following lemmas.

**Lemma 2.1.** If \( p(z) \) is analytic in \( \Delta \), \( p(0) = 1 \) and \( \text{Re} p(z) > 1/2 \), \( z \in \Delta \), then for any function \( F \) analytic in \( \Delta \), the function \( p * F \) takes its values in the convex hull of \( F(\Delta) \).

The assertion of Lemma 2.1 follows by using the Herglotz representation for \( p \). The next lemma is due to Fejér [3].

A sequence \( a_0, a_1, \ldots, a_n, \ldots \) of nonnegative numbers is called a convex null sequence if \( a_n \to 0 \) as \( n \to \infty \) and

\[
a_0 - a_1 \geq a_1 - a_2 \geq \cdots \geq a_n - a_{n+1} \geq \cdots \geq 0. \tag{2.1}
\]

**Lemma 2.2.** Let \( \{c_k\}_{k=0}^{\infty} \) be a convex null sequence. Then the function \( p(z) = c_0/2 + \sum_{k=1}^{\infty} c_k z^k \), \( z \in \Delta \), is analytic and \( \text{Re} p(z) > 0 \) in \( \Delta \).

Now we prove the following theorem.

**Theorem 2.3.**

\[
R^{n+1}(\lambda, \alpha) \subset R^n(\lambda, \alpha). \tag{2.2}
\]

**Proof.** Let \( f \) belong to \( R^{n+1}(\lambda, \alpha) \) and let it be given by (1.1). Then from (1.5), we have

\[
\text{Re} \left( \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^{n+1} a_k z^{k-1} \right) > \frac{1}{2}. \tag{2.3}
\]

Now

\[
(D^n f(z))' = 1 + \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^{n} a_k z^{k-1}
= \left( 1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k[1+(k-1)\lambda]^{n+1} a_k z^{k-1} \right)
\times \left( 1 + 2(1-\alpha) \sum_{k=2}^{\infty} \frac{z^{k-1}}{1+(k-1)\lambda} \right). \tag{2.4}
\]

Applying Lemma 2.2, with \( c_0 = 1 \) and \( c_k = 1/(1+k\lambda) \), \( k = 1, 2, \ldots \), we get

\[
\text{Re} \left( 1 + 2(1-\alpha) \sum_{k=2}^{\infty} \frac{z^{k-1}}{1+(k-1)\lambda} \right) > \alpha. \tag{2.5}
\]

Applying Lemma 2.1 to \( (D^n f(z))' \), we get the required result.
ON UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED ...

We also have a better result than Theorem 2.3.

**THEOREM 2.4.** Let \( f \in R^{n+1}(\lambda, \alpha) \). Then \( f \in R^n(\lambda, \beta) \), where

\[
\beta = \frac{2\lambda^2 + (1 + 3\lambda)\alpha}{(1 + \lambda)(1 + 2\lambda)} \geq \alpha.
\]  

**(Proof.** Let \( f \in R^{n+1}(\lambda, \alpha) \). It is shown in [9], as an example, that if \( \lambda \geq 0 \) and

\[
g(z) = z + \sum_{k=2}^{\infty} \frac{z^k}{1 + (k - 1)\lambda},
\]  

then

\[
\text{Re} \frac{g(z)}{z} > \frac{4\lambda^2 + 3\lambda + 1}{2(1 + \lambda)(1 + 2\lambda)}.
\]  

Hence

\[
\text{Re} \left( 1 + 2(1 - \alpha) \sum_{k=2}^{\infty} \frac{z^{k-1}}{1 + (k - 1)\lambda} \right) > \frac{2\lambda^2 + (1 + 3\lambda)\alpha}{(1 + \lambda)(1 + 2\lambda)}. 
\]  

Now an application of *Lemma 2.1* to \((D^nf(z))'\) in the previous theorem completes the proof. \(\square\)

**REMARK 2.5.** If we put \( n = 1 \) in *Theorem 2.4*, then we have

\[
\text{Re} \left( f'(z) + \lambda zf''(z) \right) > \alpha \Rightarrow \text{Re} f''(z) > \frac{2\lambda^2 + (1 + 3\lambda)\alpha}{(1 + \lambda)(1 + 2\lambda)},
\]  

which is an improvement of the result of Saitoh [7] for \( \lambda \geq 1 \), where he shows that, for \( \lambda > 0 \),

\[
\text{Re} \left( f'(z) + \lambda zf''(z) \right) > \alpha \Rightarrow \text{Re} f'(z) > \frac{2\alpha + \lambda}{2 + \lambda}.
\]  

Using *Theorem 2.4* \((n - m)\) times we get, after some calculations, the following theorem.

**THEOREM 2.6.** Let \( f \in R^n(\lambda, \alpha) \) and let \( n > m \geq 0 \). Then \( f \in R^m(\lambda, \beta) \) if

\[
\beta = \left[ \frac{1 + 3\lambda}{(1 + \lambda)(1 + 2\lambda)} \right]^{n-m} \alpha + \frac{2\lambda^2}{(1 + \lambda)(1 + 2\lambda)} \sum_{k=0}^{n-m-1} \left( \frac{1 + 3\lambda}{(1 + \lambda)(1 + 2\lambda)} \right)^k \geq \alpha.
\]  

If we put \( m = 0 \) in *Theorem 2.6*, we obtain the following interesting result.

**COROLLARY 2.7.** Let \( f \in R^n(\lambda, \alpha) \). Then \( \text{Re} f'(z) > \beta \), where \( \beta \) is given by (2.12) with \( m = 0 \).
**Remark 2.8.** Since $D_\lambda$ (given by (1.3)) is a linear function of $\lambda$, it is clear that

$$R^n(\lambda, \alpha) \subset R^n(\lambda', \alpha),$$  \hspace{1cm} (2.13)

where $\lambda > \lambda'$.

The following theorem deals with the partial sum of the functions in $R^n(\lambda, \alpha)$. For the proof we need the following result, due to Ahuja and Jahangiri [2].

**Lemma 2.9.** Let $-1 < t < S = 4.567802$. Then

$$\text{Re} \left( \sum_{k=2}^{m} \frac{z^{k-1}}{k + t - 1} \right) > -\frac{1}{1 + t}, \quad z \in \Delta. \hspace{1cm} (2.14)$$

**Theorem 2.10.** Let $S_m(z, f)$ denote the $m$th partial sum of a function $f$ in $R^n(\lambda, \alpha)$. If $f \in R^n(\lambda, \alpha)$ and $\lambda \geq 1/s = 0.21892$, then $S_m(z, f) \in R^{n-1}(\lambda, \beta)$, where

$$\beta = \frac{2\alpha + \lambda - 1}{\lambda + 1}. \hspace{1cm} (2.15)$$

**Proof.** Let $f \in R^n(\lambda, \alpha)$ and let it be given by (1.1). Then from (1.5) we have

$$\text{Re} \left( 1 + \sum_{k=2}^{\infty} k [1 + (k-1)\lambda] a_k z^{k-1} \right) > \alpha \hspace{1cm} (2.16)$$

or

$$\text{Re} \left( 1 + \frac{2}{\lambda + 1} \sum_{k=2}^{\infty} k [1 + (k-1)\lambda] a_k z^{k-1} \right) > \frac{2\alpha + \lambda - 1}{\lambda + 1}. \hspace{1cm} (2.17)$$

Now

$$(D^{n-1}S_m(z, f))' = 1 + \sum_{k=2}^{m} k [1 + (k-1)\lambda] a_k z^{k-1}$$

$$= \left( 1 + \frac{2}{\lambda + 1} \sum_{k=2}^{\infty} k [1 + (k-1)\lambda] a_k z^{k-1} \right)$$

$$\ast \left( 1 + \frac{\lambda + 1}{2\lambda} \sum_{k=2}^{m} \frac{z^{k-1}}{1/\lambda + (k-1)} \right), \quad \lambda > 0. \hspace{1cm} (2.18)$$

From Lemma 2.9, we see that, for $\lambda \geq 1/s = 0.21892$,

$$\text{Re} \sum_{k=2}^{m} \frac{z^{k-1}}{1/\lambda + (k-1)} > -\frac{\lambda}{\lambda + 1}, \hspace{1cm} (2.19)$$

hence

$$\text{Re} \left( 1 + \frac{\lambda + 1}{2\lambda} \sum_{k=2}^{m} \frac{z^{k-1}}{1/\lambda + (k-1)} \right) > \frac{1}{2}, \hspace{1cm} (2.20)$$

and the result follows by application of Lemma 2.1. \qed
Now we prove the following theorem.

**Theorem 2.11.** The set $R^n(\lambda, \alpha)$ is convex.

**Proof.** Let the functions

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{ki} z^k \quad (i = 1, 2) \quad (2.21)$$

be in the class $R^n(\lambda, \alpha)$. It is sufficient to show that the function $h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$, with $\mu_1$ and $\mu_2$ nonnegative and $\mu_1 + \mu_2 = 1$, is in the class $R^n(\lambda, \alpha)$.

Since

$$h(z) = z + \sum_{k=2}^{\infty} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^k, \quad (2.22)$$

then from (2.4) we have

$$(D^n h(z))' = 1 + \sum_{k=2}^{\infty} k(\mu_1 a_{k1} + \mu_2 a_{k2}) [1 + (k-1) \lambda]^{n} z^{k-1}, \quad (2.23)$$

hence

$$\text{Re}(D^n h(z))' = \text{Re}(1 + \mu_1 \sum_{k=2}^{\infty} k[1 + (k-1) \lambda]^{n} a_{k1} z^{k-1}) + \text{Re}(1 + \mu_2 \sum_{k=2}^{\infty} k[1 + (k-1) \lambda]^{n} a_{k2} z^{k-1}). \quad (2.24)$$

Since $f_1, f_2 \in R^n(\lambda, \alpha)$, this implies that

$$\text{Re}(1 + \mu_i \sum_{k=2}^{\infty} k[1 + (k-1) \lambda]^{n} a_{ki} z^{k-1}) > 1 + \mu_i(\alpha - 1) \quad (i = 1, 2). \quad (2.25)$$

Using (2.25) in (2.24), we obtain

$$\text{Re}(D^n h(z))' > 1 + \alpha(\mu_1 + \mu_2) - (\mu_1 + \mu_2), \quad (2.26)$$

and since $\mu_1 + \mu_2 = 1$, the theorem is proved.

Hallenbeck [4] showed that

$$\text{Re}f'(z) > \alpha \Rightarrow \text{Re} \frac{f'(z)}{z} > (2\alpha - 1) + 2(1 - \alpha) \log 2. \quad (2.27)$$

Using Theorem 2.3 and (2.27), we obtain the following theorem.

**Theorem 2.12.** Let $f \in R^n(\lambda, \alpha)$. Then

$$\text{Re} \frac{D^n f(z)}{z} > (2\alpha - 1) + 2(1 - \alpha) \log 2. \quad (2.28)$$

This result is sharp as can be seen by the function $f_x$ given by (3.1).
3. Extreme points. The extreme points of the closed convex hull of $R(\alpha)$ were determined by Hallenbeck [4]. We denote the closed convex hull of a family $F$ by clco$F$, and we make use of some results in [4] to determine the extreme points of $R^n(\lambda, \alpha)$.

**Theorem 3.1.** The extreme points of $R^n(\lambda, \alpha)$ are

$$f_\chi(z) = z + 2(1-\alpha) \sum_{k=2}^{\infty} \frac{x^{k-1}z^k}{k[1+(k-1)\lambda]^n}, \quad |x| = 1, \ z \in \Delta.$$  \hspace{1cm} (3.1)

**Proof.** Since $D^n : f \to D^n f$ is an isomorphism from $R^n(\lambda, \alpha)$ to $R(\alpha)$, it preserves the extreme points and, in [4], it is shown that the extreme points of $R(\alpha)$ are

$$z + 2(1-\alpha) \sum_{k=2}^{\infty} \frac{1}{k} x^{k-1}z^k, \quad |x| = 1, \ z \in \Delta.$$  \hspace{1cm} (3.2)

Hence from (1.5), we see that the extreme points of clco$R^n(\lambda, \alpha)$ are given by (3.1). Since the family $R^n(\lambda, \alpha)$ is convex (Theorem 2.6) and therefore equal to its convex hull, we get the required result. \qed

As consequences of Theorem 3.1, we have the following corollary.

**Corollary 3.2.** Let $f$ belong to $R^n(\lambda, \alpha)$ and let it be given by (1.1). Then

$$|a_k| \leq \frac{2(1-\alpha)}{k[1+(k-1)\lambda]^n}, \quad k \geq 2.$$  \hspace{1cm} (3.3)

This result is sharp as shown by the function $f_\chi(z)$ given by (3.1).

**Corollary 3.3.** If $f \in R^n(\lambda, \alpha)$, then

$$|f(z)| \leq r + \sum_{k=2}^{\infty} \frac{2(1-\alpha)}{k[1+(k-1)\lambda]^n} r^k, \quad |z| = r,$$

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty} \frac{2(1-\alpha)}{[1+(k-1)\lambda]^n} r^{k-1}, \quad |z| = r.$$  \hspace{1cm} (3.4)

This result is sharp as shown by the function $f_\chi(z)$ given by (3.1) at $z = \bar{\alpha}r$.

4. Convolution properties. Ruscheweyh and Sheil-Small [6] verified the Polya-Schoenberg conjecture and its analogous results, namely, $C * C \subset C$, $C * S^* \subset S^*$, and $C * K \subset K$, where $C$, $S^*$, and $K$ denote the classes of convex, starlike, and close-to-convex univalent functions, respectively. In the following, we prove the analogue of the Polya-Schoenberg conjecture for the class $R^n(\lambda, \alpha)$.

**Theorem 4.1.** Let $f \in R^n(\lambda, \alpha)$ and $g \in C$. Then $f * g \in R^n(\lambda, \alpha)$.

**Proof.** It is known that if $g$ is convex univalent in $\Delta$, then

$$\text{Re} \frac{g(z)}{z} > \frac{1}{2}.$$  \hspace{1cm} (4.1)
Using convolution properties, we have

\[
\text{Re}(D^n(f \ast g)(z))' = \text{Re}\left( (D^n f(z))' \ast \frac{g(z)}{z} \right),
\]

and the result follows by application of Lemma 2.1.

**Theorem 4.2.** Let \( f \) and \( g \) belong to \( R^n(\lambda, \alpha) \). Then \( f \ast g \in R^n(\lambda, \beta) \), where

\[
\beta = \frac{\lambda(2\alpha + 1) + 4\alpha - 1}{2(\lambda + 1)} \geq \alpha.
\]

**Proof.** Let \( g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in R^n(\lambda, \alpha) \), then

\[
\text{Re}\left( 1 + \sum_{k=2}^{\infty} k[1 + (k-1)\lambda]b_k z^{k-1} \right) > \alpha.
\]

Let \( c_0 = 1 \) and

\[
c_k = \frac{\lambda + 1}{(k+1)[1+k\lambda]}n, \quad k \geq 1.
\]

Then \( \{c_k\}_{k=0}^{\infty} \) is a convex null sequence. Hence, by Lemma 2.2, we have

\[
\text{Re}\left( 1 + \sum_{k=2}^{\infty} \frac{\lambda + 1}{k[1+(k-1)\lambda]}n z^{k-1} \right) > \frac{1}{2}.
\]

Now we take the convolution of (4.4) and (4.6) and apply Lemma 2.1 to obtain

\[
\text{Re}\left( 1 + (\lambda + 1) \sum_{k=2}^{\infty} b_k z^{k-1} \right) > \alpha
\]

or

\[
\text{Re}\left( \frac{g(z)}{z} \right) = \text{Re}\left( 1 + \sum_{k=2}^{\infty} b_k z^{k-1} \right) > \frac{\lambda + \alpha}{\lambda + 1}.
\]

Hence

\[
\text{Re}\left( \frac{g(z)}{z} - \frac{2\alpha + \lambda - 1}{2(\lambda + 1)} \right) > \frac{1}{2}.
\]

Since \( f \in R^n(\lambda, \alpha) \), by applying Lemma 2.1, we obtain

\[
\text{Re}\left( (D^n f(z))' \ast \left( \frac{g(z)}{z} - \frac{2\alpha + \lambda - 1}{2(\lambda + 1)} \right) \right) > \alpha \quad (4.10)
\]

or

\[
\text{Re}\left( (D^n f(z))' \ast \frac{g(z)}{z} \right) > \frac{\lambda(2\alpha + 1) + 4\alpha - 1}{2(\lambda + 1)} = \beta, \quad (4.11)
\]

and by (4.2), the result follows. \( \Box \)
 Remark 4.3. If we put $\lambda = 0$ in Theorem 4.2, we get the corresponding result for functions in $R(\alpha)$, given by Ahuja [1].

REFERENCES


F. M. Al-Oboudi: Mathematics Department, Science Sections, Girls College of Education, Sitteen Street, Malaz, Riyadh 11417, Saudi Arabia

*E-mail address:* rytelmi@gcpa.edu.sa