ON THE MAPPING \( xy \rightarrow (xy)^n \) IN AN ASSOCIATIVE RING

SCOTT J. BESLIN and AWAD ISKANDER

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We consider the following condition \((*)\) on an associative ring \( R \): \((*)\). There exists a function \( f \) from \( R \) into \( R \) such that \( f \) is a group homomorphism of \((R,+)\), \( f \) is injective on \( R \), and \( f(xy) = (xy)^{n(x,y)} \) for some positive integer \( n(x,y) > 1 \). Commutativity and structure are established for Artinian rings \( R \) satisfying \((*)\), and a counterexample is given for non-Artinian rings. The results generalize commutativity theorems found elsewhere. The case \( n(x,y) = 2 \) is examined in detail.

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Let \( R \) be an associative ring, not necessarily with unity, and let \( R^+ \) denote the additive group of \( R \). In [3], it was shown that \( R \) is commutative if it satisfies the following condition.

(I) For each \( x \) and \( y \) in \( R \), there exists \( n = n(x,y) > 1 \) such that \( (xy)^n = xy \).

We generalize this result by considering the condition below.

(II) There exists a function \( f \) from \( R \) into \( R \) such that \( f \) is a group homomorphism of \( R^+ \), \( f \) is injective on \( R \), and \( f(xy) = (xy)^{n(x,y)} \) for some positive integer \( n = n(x,y) > 1 \) depending on \( x \) and \( y \).

An example of a ring satisfying (II) for \( n(x,y) = 2 \) is given by \( R = B \oplus N \), where \( B \) is a Boolean ring and \( N \) is a zero ring (a ring with trivial product, \( xy = 0 \) for all \( x \) and \( y \)). In this case, we may take \( f \) to be the identity mapping. It was shown in [2] that a ring which is product-idempotent (i.e., \( (xy)^2 = xy \) for every \( x \) and \( y \)) must be of the form \( B \oplus N \). We will see that Artinian rings \( R \) for which (II) is true are not far removed from this structure.

In this paper, we give the structure of an Artinian ring \( R \) satisfying (II) without invoking the commutativity theorems of Bell [1]. We then exhibit an infinite noncommutative ring for which \( f \) is surjective but not injective. Throughout this paper, the notation \( J(R) \) denotes the Jacobson radical of the ring \( R \). If \( r \) is in \( R \), the symbol \( \bar{r} \) denotes the coset \( r + J(R) \).

The proposition below states that rings satisfying (II) obey the central-idempotent property.

**Proposition 1** (see [3]). Let \( R \) be a ring satisfying (II). If \( e \) is an idempotent in \( R \), then \( e \) is central.

**Proof.** Since \( f(yx) = (yx)^{n(y,x)} = y(xyx) \cdots yx \), we have that \( xy = 0 \) in \( R \) implies \( yx = 0 \), for any \( x \) and \( y \) in \( R \). Now, for every \( r \) in \( R \), \( (e^2 - e)r = e(er - r) = 0 \). Thus, \( (er - r)e = 0 \) or \( ere = re \). Similarly, \( ere = er \). Hence, \( er = re \). \( \square \)
Theorem 2. Let $R$ be an Artinian ring satisfying (II). If $(xy)^m = 0$ for some positive integer $m$, then $xy = 0$.

Proof. Suppose that $(xy)^m = 0$ and $(xy)^{m-1} 
eq 0$, $m > 1$. Then, $f[(xy)^{m-1}] = [(xy)^{m-1}]^n = 0$. Since $f$ is injective on $R^2$, $(xy)^{m-1} = 0$, a contradiction. □

Corollary 3. If $R$ is an Artinian ring satisfying (II), then $R \cdot J(R) = J(R) \cdot R = (0)$.

Proof. Since $R$ is Artinian, the ideal $J(R)$ is nilpotent. □

Corollary 4. For an Artinian ring $R$ satisfying (II), $J(R)$ is a zero ring.

Corollary 5. For an Artinian ring $R$ satisfying (II), $R/J(R)$ is commutative.

Proof. If not, there is a direct summand of $R/J(R)$ isomorphic to a full matrix ring over a division ring. Hence, there exist $\bar{u}$ and $\bar{\nu}$ in $R/J(R)$ such that $\bar{u}\bar{\nu} \neq 0$ and $\bar{u}\bar{\nu}\bar{u} = 0$. It follows that $uv \neq 0$ in $R$ and that $uvu$ is in $J(R)$. But then $f(uv) = (uv)^n(u,v) = uv\cdot uv\cdots uv = (uvu)v\cdots uv = 0$. Thus, by the injective property of $f$ on $R^2$, $uv = 0$, a contradiction.

We now obtain the structure of an Artinian ring $R$ satisfying (II). □

Theorem 6. If $R$ is an Artinian ring satisfying (II), then $R$ decomposes as a direct sum of rings $eR \oplus N$, where $e$ is an idempotent in $R$ and $N$ is a zero ring.

Proof. By Corollary 5, the ring $S = R/J(R)$ is a direct sum of fields; hence $S$ has an identity $\bar{t}$, which lifts to a central idempotent $e$ in $R$ such that $e - t$ is in $J(R)$. Let $N = \{r - er : r \in R\}$. It is easy to see that $N$ is an ideal of $R$, and that the intersection of $N$ with $eR$ is $(0)$. Clearly, $R = eR + N$, and so we may write $R = eR \oplus N$. Now, $e - t$ in $J(R)$ implies that $(e - t)^2 = 0$ or $e = 2et - t^2$. Hence, if $r$ is in $R$, $(2\bar{e} \cdot \bar{t} - \bar{t}^2)\bar{r} = \bar{e} \cdot \bar{r} = \bar{e}\bar{r}$ or $2\bar{e} \cdot \bar{t} \cdot \bar{r} - \bar{t}^2 \cdot \bar{r} = 2\bar{e} \cdot \bar{r} - \bar{r} = \bar{e}\bar{r}$, since $\bar{t}$ is the identity of $S$. Thus, $\bar{e}\bar{r} - \bar{r} = 0$ or $r - er$ is in $J(R)$. Therefore, $N$ is a zero subring of $J(R)$. □

Corollary 7. If $R$ is an Artinian ring satisfying (II), then $R$ is a direct sum $F \oplus N$, where $F$ is a direct sum of fields and $N$ is a zero ring.

Proof. By Theorem 2, the ring $eR$ in Theorem 6 has no nonzero nilpotent elements, and hence is a direct sum of fields by Corollary 5. □

Corollary 8. Let $R$ be as in Theorem 2. Then $R$ is commutative.

Corollary 9. Let $R$ be as in Theorem 2. Then $J(R)$ consists precisely of the nilpotent elements $\{x : x^2 = 0\}$.

Remark 10. The function $f$ maps the ideal $eR$ of Theorem 6 into itself, since $f(ex) = (ex)^n = e^n x^n = ex^n$.

Remark 11. The specific fields in the direct sum $F$ of Corollary 7 depend, of course, on the integers $n(x, y)$. A Boolean ring is acceptable for any value of $n$. The prime field with $p$ elements, $p$ a prime, is acceptable for $n = (p - 1)m + 1$, $m$ a positive
integer. A finite field of order $p^k$ is acceptable for $n = p$. Of course, an infinite field of characteristic $p$ need not be a $p$th root field.

We now exhibit an infinite noncommutative ring $R$ for which $f(xy) = (xy)^2$ on $R^2$.

Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Let $R$ be the free $\mathbb{Z}_4$-module with countable base $A = \{a_i : i = 1, 2, 3, \ldots\}$. On $A$, define the multiplication $a_1a_2 = a_3$, $a_2a_1 = -a_3$, $a_ia_i = 0$ otherwise. One may verify that this yields an associative multiplication which extends to a ring multiplication on $R$ considered as an abelian group. Clearly, the ring $R$ is noncommutative. Define $f : A \to A \cup \{0\}$ via $f(a_1) = f(a_3) = 0$ and $f(a_i) = a_{\rho(i)}$, $i \neq 1, 3$, where $\rho$ is any bijection of $\{2, 4, 5, \ldots\}$ onto the set of positive integers. The map $f$ extends to a group homomorphism of $R^+$. Now, $f(a_1a_i) = f(0) = 0 = (a_1a_i)^2$ for $(i, j) \neq (1, 2)$ or $(2, 1)$. Moreover, $f(a_1a_2) = f(a_3) = 0 = (a_1a_2)^2 = a_3^2$. Similarly, $f(a_2a_1) = 0 = (a_2a_1)^2$.

It is then easy to check that $f(xy) = (xy)^2$ for every $x$ and $y$ in $R$, since $a_ia_ja_k = 0$ for all $a_i, a_j, a_k$ in $A$.

The function $f$ above is not injective. We prove the following theorem which insures the commutativity of any ring $S$, given injectivity of $f$ on the subring $S^2$ alone.

**Theorem 12.** Let $f$ be a function from a ring $S$ into $S$ such that $f(x + y) = f(x) + f(y)$ and $f(xy) = (xy)^2$. Assume further that $f$ is injective on $S^2$. Then $S$ is commutative.

**Proof.** Let $x$, $y$, $z$, and $t$ be arbitrary elements of $S$. Now, $f(2xy) = 2(xy)^2 = (2xy)^2 = 4(xy)^2$, so $2(xy)^2 = f(2xy) = 0$. Hence, $2xy = 0$ by injectivity. Moreover, if $xy = 0$, then $f(xy) = y(xy)x = 0$ implies $yx = 0$. From $(xy)^2 + (yz)^2 = f(xy) + f(yz) = f((x + z)y) = [(x + z)y]^2 = (xy + yz)^2 = (xy)^2 + xyyz + zyxz + (yz)^2$, we obtain $xy = yx$. Now, $f(xtyz + yzxt) = f(xtyz) + f(yzxt) = xtyz \cdot xtyz + yzxt \cdot yzxt = (xt)y(zxt)y + yzxt \cdot yzxt = yzxt \cdot yzxt + x(ty)zt + yzxt \cdot yzxt$. Hence, $xtyz + yzxt = 0$. Thus, $(xtyz + yzxt)xtyz = xtyz \cdot xtyz + yzxt \cdot xtyz = xtyz \cdot xtyz + yzxt \cdot x(ty)zt = xtyz \cdot xtyz + yzxt(zt)xt = f(xtyz + yzxt) = 0$. Therefore, $xtyz + yzxt = 0$ or $(xt)(yz) = (yz)(xt)$. Hence, $S^2$ is commutative.

Now, $f(xy) = (xy)(yz) = (x(yz)(yz) = x(yz)^2x$. Similarly, $f(yzx) = x(yz)^2x$. So, $xyz = yzx$.

Finally, $f(xy) = (xy)(yz) = x(yzx) = x^2y^2 = y^2x^2 = (yx)(yx) = f(yx)$. Thus, $xy = yx$, and $S$ is commutative. This completes the proof.

**Remark 13.** The ring $R$ in the example preceding Theorem 12 does not have a unity. It can be shown that if $S$ is any ring in which every element is a square, and squaring is an endomorphism of $S^+$, then $S$ is commutative. It follows that a ring $R$ satisfying (II) for $n = 2$ and having a right or left identity is commutative.

In view of Remark 13 and Theorem 12, we make the following conjecture and leave it as a problem.

**Conjecture 14.** Let $S$ be a ring and $n \geq 2$ a positive integer. If the function $f(x) = x^n$ on $S$ is surjective (injective) and $f$ is a group endomorphism of $S^+$, then $S$ is commutative.
References


Scott J. Beslin: Department of Mathematics and Computer Science, Nicholls State University, Thibodaux, LA 70310, USA
E-mail address: scott.beslin@nicholls.edu

Awad Iskander: Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, USA
E-mail address: awadiskander@juno.com