ON NON-MIDPOINT LOCALLY UNIFORMLY ROTUND NORMABILITY IN BANACH SPACES

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We will show that if $X$ is a tree-complete subspace of $\ell_\infty$, which contains $c_0$, then it does not admit any weakly midpoint locally uniformly convex renorming. It follows that such a space has no equivalent Kadec renorming.

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1. Introduction. It is known that $\ell_\infty$ has an equivalent strictly convex renorming [2]; however, by a result due to Lindenstrauss, it cannot be equivalently renormed in locally uniformly convex manner [10]. In this note, we will show that every tree-complete subspace of $\ell_\infty$, which contains $c_0$, does not admit any equivalent midpoint locally uniformly convex norm. This can be considered as an extension of [1, 8]. Since every strictly convexifiable Banach space with Kadec property admits an equivalent midpoint locally uniformly convex renorming [9], it follows that every subspace of $\ell_\infty$ with the tree-completeness property has no equivalent Kadec renorming. The existence of such a (nontrivial) subspace, which does not contain any copy of $\ell_\infty$, has already been proved by Haydon and Zizler (see [5, 7]).

2. Results. We recall that a norm $\| \cdot \|$ on a Banach space $X$ is said to be midpoint locally uniformly rotund (MLUR) if, whenever $\{x_n\}, \{y_n\}$, and $x$ are in $X$ with $\|x_n\| \to \|x\|$, $\|y_n\| \to \|x\|$, and $\|(x_n + y_n)/2 - x\| \to 0$, we necessarily have $\|x_n - y_n\| \to 0$. If at the end of the last sentence, we replace norm with weak, the definition of weakly midpoint locally uniformly rotund (wMLUR) will be obtained [3]. Let $T$ be the set of all finite (possible empty) strings of 0’s and 1’s. The empty string () is the unique string of length 0; the length $|t|$ of a string $t$ is $n$ if $t \in \{0, 1\}^n$. The tree order is defined by $s \prec t$ if $|s| < |t|$ and $t(m) = s(m)$ for $m \leq |s|$. Each $t \in T$ has exactly two immediate successors, that is, $t0$ and $t1$.

A lattice $L$ is said to be tree-complete if, whenever $\{f_t\}_{t \in T}$ is a bounded disjoint family in $L$, there exists $b \in \{0, 1\}^N$, such that $\sum_{n \in N} f(b|n)$ is in $L$.

Haydon and Zizler [7] constructed a closed linear subspace of $\ell_\infty$ (which is a tree-complete sublattice of $\ell_\infty$) such that it contains $c_0$ but does not contain any subspace isomorphic to $\ell_\infty$. Notice that in this space $X$ every infinite subset $M$ of $N$ has an infinite subset $M_0 \subset M$ such that $1_{M_0} \in X$ [7].

**Theorem 2.1.** Let $X$ be a tree-complete sublattice of $\ell_\infty$. If $X$ contains $c_0$, then $X$ does not admit any equivalent wMLUR renorming.
for each
\[ g_k \alpha(\cdot) \]
with the following properties.

Choose an element \( f_i(\cdot) \) of \( X \) such that \( |||f_i(\cdot)||| > (3M_i + m_i)/4 \). Then select two disjoint infinite subsets \( N_i' \) and \( N_i'' \) of \( N_i \setminus \text{supp}(f_i) \) with \( 1_{N_i'} \in X \) for some \( k_i \in N_i' \), define \( N_i = N_i'' \setminus \{k_i\} \), and let
\[ A_i = \{ f \in A_i : f(n) = f_i(n) \text{ for each } n \notin N_i \} \quad (i = 0, 1). \]

Suppose that for some \( t \in T \), with \( |t| < n \), \( A_t \) is specified. Put
\[ M_t = \sup \{|f(\cdot)| : f \in A_t\}, \quad m_t = \inf \{|f(\cdot)| : f \in A_t\}. \]

Let \( f_t \in A_t \) satisfy \( |||f_t(\cdot)||| > (3M_t + m_t)/4 \) and take two disjoint infinite subsets \( N_t' \) and \( N_t'' \) of \( N_t \setminus \text{supp}(f_t) \) with \( 1_{N_t'} \in X \), put \( N_t = N_t'' \setminus \{k_t\} \), and define
\[ A_{ti} = \{ f \in A_t : f(n) = f_i(n) \text{ for each } n \notin N_{ti} \} \quad (i = 0, 1). \]

Thus, by induction on \(|t|\), we can obtain a family \( \{A_t\}_{t \in T} \) of subsets of \( X \), a family \( \{f_t\} \) of elements of \( X \), a family \( \{N_t\} \) of infinite subsets of \( N \), and a family of integers \( \{k_t\} \) with the following properties.

(a) \( A_{ti} \) is of the form
\[ A_{ti} = \{ f \in A_t : f(n) = f_i(n) \text{ for each } n \notin N_{ti} \} \quad (i = 0, 1), \]
for each \( t \in T \).

(b) \( k_{ti} \in N_t \setminus N_{ti} \) and \( f_t(k_{ti}) = 0 \) for \( t \in T \) and \( i = 0, 1 \).

(c) \(|f_t(\cdot)| > (3M_t + m_t)/4 \), where \( M_t \) and \( m_t \) denote the supremum and infimum of \( \{|||f(\cdot)||| : f \in A_t\} \), respectively.

(d) \( N_s \subset N_t \) whenever \( t < s \) and \( N_t \cap N_s = \emptyset \), if \( s \) and \( t \) are not comparable.

(e) \( \text{supp}(f_t - f_s) \subset N_t \setminus N_s \) for \( t < s \).

By (e), \( \{g_t\}_{t \in T} \), defined by
\[ g(\cdot) = f(\cdot), \quad g_{ti} = f_{ti} - f_t \quad (i = 0, 1), \]
is a disjoint family of elements of \( X \). By the tree-completeness of \( X \), there exists some \( b \in \{0, 1\}^N \) such that
\[ f_b(x) = f(\cdot) + \sum_{n \in N} g_b|n \quad (2.7) \]
is in \( X \). Let \( \{k_{\alpha(n)}\} \) be a subsequence of \( \{k_{b|n}\} \) such that \( 1_E \in X \), where \( E = \{k_{\alpha(1)}, \ldots\} \). Let \( E_n = \{k_{\alpha(n)}, k_{\alpha(n+1)}, \ldots\} \) and \( h_n = 1_{E_n} \). By (a) and (b), \( g_{n+1} = f_b + h_{n+1} \) and \( g_{n+1} = f_b - h_{n+1} \) are in \( A_{b|n} \). Next, select some \( \mu \in X^* \), such that \( \mu(h_1) = 1 \) and \( \mu(g) = 0 \) for each \( g \in c_0 \). Clearly, for such an element \( \mu \) and each \( n \in N \), we have \( \mu(h_n) = 1 \). By
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(a), $2 f_b - f \in A_{b|n}$, thus $|||2 f_b||| - |||f||| \leq M_{b|n}$ for each $f \in A_{b|n}$ and $n \in N$. It follows that
\[
\frac{(3M_{b|n-1} + m_{b|n-1})}{2} \leq |||2 f_b||| \leq M_{b|n} + |||f|||, \quad \forall f \in A_{b|n},
\]  
and so
\[
\frac{(3M_{b|n-1} + m_{b|n-1})}{2} \leq M_{b|n} + m_{b|n} \leq M_{b|n-1} + m_{b|n-1}, \quad \forall n \in N.
\]  
Therefore,
\[
M_{b|n} - m_{b|n} \leq M_{b|n} - \frac{(M_{b|n-1} + m_{b|n-1})}{2} \leq M_{b|n-1} - \frac{(M_{b|n-1} + m_{b|n-1})}{2} = \frac{(M_{b|n-1} - m_{b|n-1})}{2}.
\]  

The above relations show that
\[
|||g^+_{n+1}||| - |||f_b||| \leq M_{b|n} - m_{b|n} \leq \frac{(M_{b|n-1} - m_{b|n-1})}{2} \leq \frac{(M_{(1)} - m_{(1)})}{2n}.
\]  
That is \( \lim |||g^+_{n}||| = |||f_b||| = \lim |||g^-_{n}||| \). Moreover, \( f_b = (g^+_{n} + g^-_{n})/2 \). But \( \text{weak-lim}(g^+_{n} - g^-_{n}) \neq 0 \), since \( \mu(h_n) = 1 \) for each \( n \in N \). This shows that \( X \) does not admit any wMLUR norm.\]

It is known that weakly midpoint locally uniformly rotundity of a Banach space \( X \) is equivalent to saying that every point of \( S(\hat{X}) \) is an extreme point of \( B(X^{**}) \) \( [11] \). It follows that the space considered in Theorem 2.1 has no equivalent norm such that \( S(\hat{X}) \) is a subset of \( B(X^{**}) \).

A norm on a Banach space \( X \) is said to be strictly convex (rotund) (R) if the unit sphere of \( X \) contains no nontrivial line segment. We say that a norm is Kadec if the weak and norm topologies coincide on the unit sphere. Every MLUR Banach space admits Kadec renorming (see \( [1] \)). Haydon in \( [6, \text{Corollary 6.6}] \) gives an example of a Kadec renormable space which has no equivalent R norm. The following result gives an example of a strictly convexifiable space with no equivalent Kadec norm.

**Corollary 2.2.** If a tree-complete subspace \( X \) of \( \ell_\infty \) contains \( c_0 \), then it does not admit any equivalent Kadec renorming.

**Proof.** It is known that \( \ell_\infty \) admits an equivalent strictly convex norm (see \( [4, \text{page 120}] \) or \( [2] \)). In \( [9] \) it is shown that every R Banach space with the Kadec property admits an equivalent MLUR renorming (see also \( [3, \text{chapter IV}] \)). Thus the result follows from Theorem 2.1.\]

\( \square \)
References


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