A REMARK ON THE INTERSECTION OF THE CONJUGATES
OF THE BASE OF QUASI-HNN GROUPS

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Quasi-HNN groups can be characterized as a generalization of HNN groups. In this paper, we show that if $G^*$ is a quasi-HNN group of base $G$, then either any two conjugates of $G$ are identical or their intersection is contained in a conjugate of an associated subgroup of $G$.

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1. Introduction. In [8, Lemma 3.15, page 152], Scott and Wall proved that if $G = G_1 *_{C} G_2$ is a nontrivial free product with amalgamation group, then either $gG_1 g^{-1} \cap G_i$ is a subgroup of a conjugate of $C$, or $i = 1$ and $g \in G_1$, so that $gG_1 g^{-1} \cap G_1 = G_1$. In this paper we generalize such a result to groups acting on trees with inversions and then apply the result we obtain to a new class of groups called quasi-HNN groups, introduced in [2]. This paper is divided into five sections. In Section 2, we give basic definitions. In Section 3, we have notations related to groups acting on trees with inversions. In Section 4, we discuss the intersections of vertex stabilizers of groups acting on trees with inversions. In Section 5, we apply the results of Section 4 to a tree product of groups and of quasi-HNN groups.

2. Groups acting on graphs. In this section, we begin by recalling some definitions taken from [3, 7]. First we give formal definitions related to groups acting on graphs with inversions. By a graph $X$ we understand a pair of disjoint sets $V(X)$ called the set of vertices and $E(X)$ called the set of edges, with $V(X)$ nonempty, equipped with two maps $E(X) \to V(X) \times V(X)$, $y \to (o(y),t(y))$, and $E(X) \to E(X)$, $y \to \overline{y}$, satisfying the conditions $\overline{\overline{y}} = y$ and $o(\overline{y}) = t(y)$ for all $y \in E(X)$. The case $\overline{\overline{y}} = y$ is possible for some $y \in E(X)$. For $y \in E(X)$, $o(y)$ and $t(y)$ are called the ends of $y$ and $\overline{y}$ is called the inverse of $y$. There are obvious definitions of trees, morphisms of graphs, and $\text{Aut}(X)$, the set of all automorphisms of the graph $X$ which is a group under the composition of morphisms. We say that a group $G$ acts on a graph $X$ if there is a group homomorphism $\phi : G \to \text{Aut}(X)$. If $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. Thus if $g \in G$ and $y \in E(X)$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, and $g(\overline{y}) = \overline{g(y)}$. The case $g(y) = \overline{y}$ for some $g \in G$ and $y \in E(X)$ may occur. That is, $G$ acts with inversions on $X$.

We have the following definitions related to the action of the group $G$ on the graph $X$.

1. If $x \in X$ (vertex or edge), define $G(x)$ to be the set $G(x) = \{ g(x) : g \in G \}$. This set is called the orbit that contains $x$. 
(2) If \( x, y \in X \), define \( G(x, y) \) to be the set \( G(x, y) = \{ g \in G : g(x) = y \} \), and \( G(x, x) = G_x \), the stabilizer of \( x \). Thus, \( G(x, y) \neq \emptyset \) if and only if \( x \) and \( y \) are in the same orbit. If \( y \in E(X) \) and \( u \in \{ o(y), t(y) \} \), then it is clear that \( G_u = G_y \) and \( G_y \leq G_u \).

(3) If \( X \) is connected, then a subtree \( T \) of \( X \) is called a tree of representatives for the action of the group \( G \) on \( X \) if \( T \) contains exactly one vertex from each vertex orbit, and the subgraph \( Y \) of \( X \) containing \( T \) is called a fundamental domain if each edge of \( Y \) has at least one end in \( T \), and \( Y \) contains exactly one edge \( y \) from each edge orbit such that \( G(y, \bar{y}) = \emptyset \), and exactly one pair \( x, \bar{x} \) from each edge orbit such that \( G(x, \bar{x}) \neq \emptyset \).

3. Notations. Let \( G \) be a group acting on a tree \( X \) with inversions, let \( T \) be a tree of representatives for the action of \( G \) on \( X \), and let \( Y \) be a fundamental domain. We have the following notations.

(1) For any vertex \( v \) of \( X \), let \( v^* \) be the unique vertex of \( T \) such that \( G(v, v^*) \neq \emptyset \). That is, \( v \) and \( v^* \) are in the same vertex orbit.

(2) For each edge \( y \) of \( Y \), define the following:
   - \( [y] \) is an element of \( G(t(y), (t(y))^*) \). That is, \( [y]((t(y))^*) = t(y) \) is chosen as follows:
     - if \( o(y) \in V(T) \), then \( [y] = 1 \) in case \( y \in E(T) \), and \( [y](y) = \bar{y} \) if \( G(y, \bar{y}) \neq \emptyset \),
     - if \( o(y) \notin V(T) \), then \( [y] = [\bar{y}]^{-1} \) if \( G(y, \bar{y}) = \emptyset \), otherwise \( [y] = [\bar{y}] \) if \( G(y, \bar{y}) \neq \emptyset \);
   - \( -y \) is the edge \( -y = [y]^{-1}(y) \) if \( o(y) \in V(T) \), otherwise \( -y = y \);
   - \( +y \) is the edge \( +y = [y](-y) \). It is clear that \( t(-y) = (t(y))^* \), \( o(+y) = (o(y))^* \), \( G_{-y} \leq G_{(t(y))^*} \), \( (\bar{y}) = +(\bar{y}) \), and \( G_{+y} \leq G_{o(y)} \). Moreover, if \( G(y, \bar{y}) \neq \emptyset \), or \( y \in E(T) \), then \( G_{-y} = G_{+y} = G_y \);
   - \( \phi_y \) is the map \( \phi_y : G_{-y} \to G_{+y} \) given by \( \phi_y(g) = [y]g[y]^{-1} \);
   - \( \delta_y \) is the element \( \delta_y = [y][\bar{y}] \). It is clear that \( \phi_y \) is an isomorphism and \( \delta_y = 1 \) if \( G(y, \bar{y}) = \emptyset \). Otherwise \( \delta_y = [y]^2 \).

4. On the intersection of vertex stabilizers of groups acting on trees with inversions. In this section, \( G \) will be a group acting on a tree \( X \) with inversions, \( T \) is a tree of representatives for the action of \( G \) on \( X \), and \( Y \) is a fundamental domain. We have the following definition.

**Definition 4.1.** A word \( w \) of \( G \) means an expression of the form \( w = g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdots \cdot y_n \cdot g_n \), \( n \geq 0 \), \( y_i \in E(Y) \), for \( i = 1, \ldots, n \), such that

1. \( g_0 \in G_{o(y_1)}^* \),
2. \( g_i \in G_{(t(y_i))^*} \) for \( i = 1, \ldots, n \),
3. \( (t(y_i))^* = (o(y_i))^* \) for \( i = 1, \ldots, n - 1 \).

Define \( o(w) = (o(y_1))^* \) and \( t(w) = (t(y_n))^* \).

If \( o(w) = t(w) \), then \( w \) is called a closed word of \( G \) of type \( v \), \( v = o(w) \).

The following concepts are related to the word \( w \) defined above:

1. \( n \) is called the length of \( w \) and is denoted by \( |w| = n \),

(ii) \( w \) is called a trivial word of \( G \) if \( |w| = 0 \) (or \( w = g_0 \)),

(iii) the value of \( w \), denoted by \([w]\), is defined to be the element of \( G \):

\[
[w] = g_0[y_1]g_1[y_2]g_2 \cdots [y_n]g_n
\]  

(4.1)

(iv) the inverse of \( w \), denoted by \( w^{-1} \), is defined to be the word of \( G \):

\[
w^{-1} = g_n^{-1} \cdot \bar{y}_n \cdot \delta_{y_n}^{-1} \cdot g_{n-1}^{-1} \cdot \cdots \cdot g_2^{-1} \cdot \bar{y}_2 \cdot \delta_{y_2}^{-1} \cdot g_1^{-1} \cdot \bar{y}_1 \cdot \delta_{y_1}^{-1} \cdot g_0^{-1},
\]  

(4.2)

(v) \( w \) is called reduced if \( w \) contains no subword of the form \( y_i \cdot g_i \cdot y_i \) if \( g_i \in G_{y_i} \),

\[\text{or } y_i \cdot g_i \cdot y_i \text{ if } g_i \notin G_{y_i} \text{ if } G(y_i, y_i) \neq \emptyset \text{ for } i = 1, \ldots, n.\]

**Lemma 4.2.** Let \( w \) be a nontrivial reduced word of \( G \) and let \( a \in G_{o(w)} \) be such that \([w]^{-1} a[w] \in G_{|w|(t(w))}\). Then there exists a reduced path \( x_1, \ldots, x_n \) in \( X \) from \( o(w) \) to \([w](t(w))\) such that \( a \in G_{x_i} \) for \( i = 1, \ldots, n.\)

**Proof.** Let \( w = g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \cdots \cdot y_n \cdot g_n \), \( n \geq 1 \). By assumption, \([w]^{-1} a[w] = b\), where \( b \in G_{|w|(t(w))}\). Consider the word

\[
w_0 = g_n^{-1} \cdot \bar{y}_n \cdot \delta_{y_n}^{-1} \cdot g_{n-1}^{-1} \cdot \cdots \cdot g_2^{-1} \cdot \bar{y}_2 \cdot \delta_{y_2}^{-1} \cdot g_1^{-1} \cdot \bar{y}_1 \cdot \delta_{y_1}^{-1} \cdot g_0^{-1} \cdot a \cdot y_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \cdots \cdot y_n \cdot g_n \cdot b^{-1}.
\]  

(4.3)

Then \( w_0 \) is a nontrivial closed word of \( G \) such that \([w_0] = 1\), the identity element of \( G \). Therefore by [4, Corollary 1], \( w_0 \) is not reduced. Since \( w \) is reduced, then \( w^{-1} \) is reduced. Therefore the only possibility that makes \( w_0 \) not reduced is \( L_i^{-1} a L_i \in G_{y_i} \), where \( L_i = g_0[y_1]g_1[y_2]g_2 \cdots [y_i-1]g_{i-1} \) for \( i = 1, \ldots, n \) with the convention that \([y_0] = 1\). Then \( a \in L_i G_{y_i} L_i^{-1} = G_{y_i} L_i \) for \( i = 1, \ldots, n \). By taking \( x_i = L_i y_i \), we see that \( a \in G_{x_i} \) for \( i = 1, \ldots, n \). By the corollary of [5, Theorem 1], \( x_1, \ldots, x_n \) is a reduced path in \( X \) from \( o(w) \) to \([w](t(w))\). This completes the proof.

**Theorem 4.3.** For any two vertices \( u \) and \( v \) of \( X \), \( G_u = G_v \) or \( G_u \cap G_v \) is contained in \( G_x \), where \( x \) is an edge in the reduced path in \( X \) joining \( u \) and \( v \).

**Proof.** If \( G_u = G_v \), we are done. Let \( G_u \neq G_v \) and \( h \in G_u \cap G_v \). Then it is clear that \( u \neq v \). We need to show that \( h \) is in \( G_x \), where \( x \) is an edge in the reduced path in \( X \) joining \( u \) and \( v \). We have \( u = f(u^*) \) and \( v = g(v^*) \), where \( f \) and \( g \) are in \( G \) and \( u^* \) and \( v^* \) are the unique vertices of \( T \) such that \( G(u, u^*) \neq \emptyset \) and \( G(v, v^*) \neq \emptyset \).

Then \( h = f a f^{-1} = g b g^{-1} \), where \( a \in G_{u^*} \) and \( b \in G_{v^*} \). By [5, Lemma 2], there exists a reduced word \( w = g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \cdots \cdot y_n \cdot g_n \) of \( G \) such that \( o(w) = u, t(w) = v, \) and \([w] = g \cdot w \) is nontrivial. For, if \( w \) is trivial, then \( u^* = v^* \) and \( f^{-1} g \in G_{u^*} \). This implies that \( f^{-1} g(v^*) = u^* \), or equivalently \( u = v \). This contradicts the assumption that \( u \neq v \). By Lemma 4.2, there exists a reduced path \( p_1, \ldots, p_n \) in \( X \) joining \( o(w) = u^* \) and \([w](t(w)) \) such that \( a \in G_{p_i} \) for \( i = 1, \ldots, n \). Let \( x_i = f(p_i), i = 1, \ldots, n \). Then it is clear that \( x_1, \ldots, x_n \) is the reduced path in \( X \) joining \( u \) and \( v \) and \( h \in G_{x_i} \) for \( i = 1, \ldots, n \). This implies that \( G_u \cap G_v \leq G_{x_i} \) for \( i = 1, \ldots, n \). This completes the proof.
We have the following corollaries of Theorem 4.3.

**Corollary 4.4.** For any edge $x$ of $X$, $G_{o(x)} = G_{t(x)}$ or $G_{o(x)} \cap G_{t(x)} = G_x$.

**Corollary 4.5.** Let $u$ and $v$ be two vertices of $X$ and let $x_1, \ldots, x_n$ be the reduced path in $X$ joining $u$ and $v$ such that $G_u \neq G_v$. Then $G_u \cap G_v \leq I_{i=1}^n G_{x_i}$.

**Corollary 4.6.** Let $u$ and $v$ be two vertices of $X$ such that $G_u \neq G_v$ and let $x$ be an edge in the reduced path in $X$ joining $u$ and $v$. Then $G_u \cap G_v \leq G_x$.

**Corollary 4.7.** Let $u$ be a vertex of $X$ and let $v$ be a vertex of $T$. Then $G_u \cap G_v \leq G_x$, where $x$ is an edge in the reduced path in $X$ joining $u$ and $v$, or $u^* = v$ and $G_u \cap G_v = G_v$.

**Corollary 4.8.** Let $u$ be a vertex of $X$. Then $G_u \cap G_u^* \leq G_x$, where $x$ is an edge in the reduced path in $X$ joining $u$ and $u^*$, or $u^* = u$ and $G_u \cap G_u^* = G_u$.

**Corollary 4.9.** For any edge $y$ of $Y$, $G_{(o(y))}^* = G_{(t(y))}^*$, or $G_{(o(y))}^* \cap G_{(t(y))}^* \leq G_m$, where $m$ is an edge in the reduced path in $T$ joining $(o(y))^*$ and $(t(y))^*$.

5. Applications. In this section Theorem 4.3 and its corollaries are applied to a non-trivial tree product of groups introduced in [1] and of quasi-HNN groups introduced in [2].

In [5, Lemma 8], Mahmood showed that if $G = \prod_{i \in I} (A_i, U_{jk} = U_{kj})$ is a nontrivial tree product of the groups $A_i$, $i \in I$, then there exists a tree $X$ on which $G$ acts without inversions such that any tree of representatives for the action of $G$ on $X$ equals the fundamental domain and for every vertex $u$ of $X$ and every edge $x$ of $X$, $G_u$ is a conjugate of $A_i$ for some $i$ in $I$ and $G_x$ is a conjugate of $U_{ik}$ for some $i, k$ in $I$.

In [6, Lemma 5.1], Mahmood and Khanfar showed that if $G^*$ is the quasi-HNN group $G^* = \langle A, t_i, t_j | \rel G, t_i A_i t_i^{-1} = B_i, t_j C_j t_j^{-1} = C_j, t_j^2 = c_j, i \in I, j \in J \rangle$, then there exists a tree $X$ on which $G^*$ acts with inversions such that $G^*$ is transitive on $V(X)$ and for every vertex $v$ of $X$ and every edge $x$ of $X$, $G_v^*$ is a conjugate of $G$ and $G_v^*$ is a conjugate of $A_i$, $i \in I$, or a conjugate of $C_j$, $j \in J$.

Then by Theorem 4.3, the following two propositions hold.

**Proposition 5.1.** Let $G = \prod_{i \in I} (A_i, U_{jk} = U_{kj})$ be a nontrivial tree product of the groups $A_i$, $i \in I$. Then for any $g$ in $G$ and $i$ and $j$ in $I$, either $g A_i g^{-1} \cap A_j$ is contained in a conjugate of $U_{ij}$ or $i = j$, $g \in A_i$, and $g A_i g^{-1} \cap A_i = A_i$. Moreover, if $A_i$ and $A_j$ are adjacent, then $A_i \cap A_j = U_{ij}$.

**Proposition 5.2.** Let $G^*$ be the quasi-HNN group

\[
G^* = \langle G, t_i, t_j | \rel G, t_i A_i t_i^{-1} = B_i, t_j C_j t_j^{-1} = C_j, t_j^2 = c_j, i \in I, j \in J \rangle.
\]  

(5.1)

Then for any $g \in G^*$, $g G g^{-1} \cap G$ is contained either in a conjugate of $A_i$, $i \in I$, or in a conjugate of $C_j$, $j \in J$, or $g \in G$ and $g G g^{-1} \cap G = G$.

**Remark 5.3.** If $J = \emptyset$, then $G^*$ is the HNN group $G^* = \langle G, t_i | \rel G, t_i A_i t_i^{-1} = B_i, i \in I \rangle$. Then, for any $g \in G^*$, either $g G g^{-1} \cap G$ is contained in a conjugate of $A_i$, $i \in I$, or $g \in G$ and $g G g^{-1} \cap G = G$. 


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