Let $h$ be an entire function and $T_h$ a differential operator defined by $T_h f = f' + hf$. We show that $T_h$ has the Hyers-Ulam stability if and only if $h$ is a nonzero constant. We also consider Ger-type stability problem for $|1 - f'/hf| \leq \varepsilon$.

2000 Mathematics Subject Classification: 34K20, 26D10.

1. Introduction. The first result, which we now call the Hyers-Ulam stability (HUS), is due to Hyers [4] who gave an answer to a question posed by Ulam (cf. [11, Chapter VI] and [12]) in 1940 concerning the stability of homomorphisms: for what metric groups $G$ is it true that an $\varepsilon$-automorphism of $G$ is necessarily near to a strict automorphism?

An answer to the above problem has been given as follows. Suppose $E_1$ and $E_2$ are two real Banach spaces and $f : E_1 \to E_2$ is a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$, the set of all real numbers, for each fixed $x \in E_1$. If there exist $\theta \geq 0$ and $p \in \mathbb{R} \setminus \{1\}$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta (\|x\|^p + \|y\|^p)$$

(1.1)

for all $x, y \in E_1$, then there is a unique linear mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq 2\theta \|x\|^p / (2 - 2^p)$ for every $x \in E_1$. Hyers [4] obtained the result for $p = 0$. Then Rassias [7] generalized the above result of Hyers to the case where $0 \leq p < 1$, while the proof given in [7] also works for $p < 0$. Gajda [2] solved the problem for $1 < p$ and also gave an example that a similar result does not hold for $p = 1$ (cf. [8]).

In connection with the stability of exponential functions, Alsina and Ger [1] remarked that the differential equation $y' = y$ has the HUS. More explicitly, suppose $I$ is an open interval, $\varepsilon > 0$, and $f : I \to \mathbb{R}$ is a differentiable function such that $|f'(t) - f(t)| \leq \varepsilon$ for all $t \in I$. Then, there is a differentiable function $g : I \to \mathbb{R}$ such that $g' = g$ and $|f(t) - g(t)| \leq 3\varepsilon$ for all $t \in I$. The third and first authors of this paper along with Miyajima [10] considered the Banach-space-valued differential equation $y' = \lambda y$, where $\lambda$ is a complex constant. Then they proved the HUS of $y' = \lambda y$ under the condition that $\Re \lambda \neq 0$. Though, they treated the result as the stability of the operator $D - I_d$, where $D$ denotes the ordinary differential operator and $I_d$ the identity. Some stability results of other differential equations (or operators) are also known (cf. [5, 6, 9]).
Taking the group structure of $\mathbb{C} \setminus \{0\}$ into account, Ger and Šemrl [3] considered the inequality

$$\left| \frac{f(x + y)}{f(x)f(y)} - 1 \right| \leq \theta \quad (x, y \in S) \quad (1.2)$$

for a mapping $f : S \to \mathbb{C} \setminus \{0\}$, where $(S, +)$ is a semigroup and $\mathbb{C}$ is the set of all complex numbers. If $0 \leq \theta < 1$ and if $(S, +)$ is a cancellative abelian semigroup, then they proved that there is a unique function $g : S \to \mathbb{C} \setminus \{0\}$ such that $g(x + y) = g(x)g(y)$ for all $x, y \in S$ and that

$$\max \left\{ \left| \frac{f(x)}{g(x)} - 1 \right|, \left| \frac{g(x)}{f(x)} - 1 \right| \right\} \leq \sqrt{1 + \frac{1}{\theta^2}} - 2 \sqrt{1 + \frac{1}{\theta}} \quad (1.3)$$

for all $x \in S$. The stability phenomena of this kind is called Ger-type stability.

Throughout this paper, $H(\mathbb{C})$ stands for the set of all entire functions. Let $h \in H(\mathbb{C})$ and $T_h : H(\mathbb{C}) \to H(\mathbb{C})$ be a linear differential operator defined by

$$T_h f(z) = f'(z) + h(z)f(z) \quad (f \in H(\mathbb{C}), \ z \in \mathbb{C}). \quad (1.4)$$

**Definition 1.1.** The operator $T_h$ is said to have the HUS if and only if there exists a constant $K \geq 0$ with the following property: to each $\varepsilon \geq 0$ and $f, g \in H(\mathbb{C})$ satisfying $\sup_{z \in \mathbb{C}} |T_h f(z) - g(z)| \leq \varepsilon$, there exists an $f_0 \in H(\mathbb{C})$ such that $T_h f_0 = g$ and $\sup_{z \in \mathbb{C}} |f(z) - f_0(z)| \leq K \varepsilon$. Such $K$ is called an HUS constant for $T_h$. If, in addition, the minimum of all such $K$’s exists, then it is called the HUS constant for $T_h$.

In this paper, we first consider the HUS of the differential operator $T_h$. Then we show that $T_h$ has the HUS if and only if $h \in H(\mathbb{C})$ is a nonzero constant function. Moreover, we give the HUS constant for $T_h$. Finally, we consider the Ger-type stability problem of the differential equation $y' = \lambda y$. To be more explicit, suppose $\varepsilon \geq 0$ and $f \in H(\mathbb{C})$ satisfies

$$\sup_{z \in \mathbb{C}} \left| \frac{f'(z)}{\lambda f(z)} - 1 \right| \leq \varepsilon. \quad (1.5)$$

Does there exist $K \geq 0$ such that

$$\sup_{z \in \mathbb{C}} \left| \frac{f(z)}{ce^{\lambda t}} - 1 \right| \leq K \varepsilon \quad \text{or} \quad \sup_{z \in \mathbb{C}} \left| \frac{ce^{\lambda t}}{f(z)} - 1 \right| \leq K \varepsilon \quad (1.6)$$

holds for some $c \in \mathbb{C} \setminus \{0\}$? To this problem, we give a negative answer: the Ger-type stability does not hold in general. Moreover, we show that the solution $f \in H(\mathbb{C})$ to the differential equation $y' = \lambda y$ is only the function which satisfies both (1.5) and (1.6).

### 2. The HUS for $T_h$.

For simplicity, we write $\int_0^z f(\xi) d\xi$ for $\int_0^1 f(zt) z dt$, where $z \in \mathbb{C}$ and $f \in H(\mathbb{C})$. We associate to each $h \in H(\mathbb{C})$ a function $\tilde{h}$ defined by

$$\tilde{h}(z) = \exp \int_0^z h(\zeta) d\zeta \quad (z \in \mathbb{C}). \quad (2.1)$$
Let $h \in H(\mathbb{C})$. Throughout this section, $T_h : H(\mathbb{C}) \to H(\mathbb{C})$ denotes a linear differential operator defined by (1.4). Suppose $f, g \in H(\mathbb{C})$. Then note that $T_h f = g$ if and only if $f$ is of the form

$$f(z) = \frac{1}{\tilde{h}(z)}\left\{f(0) + \int_0^z g(\zeta)\tilde{h}(\zeta)d\zeta\right\} \quad (z \in \mathbb{C}). \quad (2.2)$$

**Lemma 2.1.** Suppose $h \in H(\mathbb{C})$ is not a constant function, $f \in H(\mathbb{C})$, and

$$0 < \sup_{z \in \mathbb{C}} |T_h f(z)| < \infty. \quad (2.3)$$

Then

$$\sup_{z \in \mathbb{C}} \left| f(z) - \frac{c}{\tilde{h}(z)} \right| = \infty \quad (2.4)$$

for every $c \in \mathbb{C}$.

**Proof.** By hypothesis, $T_h f$ is a bounded entire function, and so $T_h f$ must be constant, say $c_0 \in \mathbb{C} \setminus \{0\}$ by Liouville’s theorem. Hence, by (2.2), $f$ is of the form

$$f(z) = \frac{1}{\tilde{h}(z)}\left\{f(0) + c_0 \int_0^z \tilde{h}(\zeta)d\zeta\right\} \quad (z \in \mathbb{C}). \quad (2.5)$$

Suppose $\sup_{z \in \mathbb{C}} |f(z) - c_1/\tilde{h}(z)| < \infty$ for some $c_1 \in \mathbb{C}$. Another application of Liouville’s theorem yields the existence of a constant $c_2 \in \mathbb{C}$ such that $c_2 = f - c_1/\tilde{h}$, and therefore (2.5) gives

$$c_2\tilde{h}(z) = f(0) - c_1 + c_0 \int_0^z \tilde{h}(\zeta)d\zeta \quad (z \in \mathbb{C}). \quad (2.6)$$

By differentiating both sides of (2.6) with respect to $z$, we obtain

$$c_2 \tilde{h} = c_0 \tilde{h}, \quad (2.7)$$

and hence

$$c_2 \tilde{h} = c_0. \quad (2.8)$$

Since $h$ is not constant, this implies that $c_2 = 0$. Thus, $f = c_1/\tilde{h}$, and hence $T_h f = 0$ (see (2.2)), which contradicts $0 < \sup_{z \in \mathbb{C}} |T_h f(z)|$. \hfill $\square$

**Theorem 2.2.** If $h \in H(\mathbb{C})$, then each of the following statements implies the other:

(a) $h$ is a nonzero constant function,

(b) $T_h$ has the HUS.

**Proof.** (a)$\Rightarrow$(b). Suppose $h$ is a nonzero constant function, say $\lambda \in \mathbb{C} \setminus \{0\}$. Then, $\tilde{h}(z) = e^{\lambda z}$ for $z \in \mathbb{C}$. Suppose $\varepsilon \geq 0$ and $f, g \in H(\mathbb{C})$ satisfy $\sup_{z \in \mathbb{C}} |T_h f(z) - g(z)| \leq \varepsilon$. Then there exists a $c_0 \in \mathbb{C}$ such that $T_h f - g = c_0$ by Liouville’s theorem. Put

$$u(z) = e^{-\lambda z}\left\{\int_0^z g(\zeta)e^{\lambda \zeta}d\zeta\right\} \quad (z \in \mathbb{C}). \quad (2.9)$$
Then $T_h u = g$, and so $T_h (f - u) = c_0$, $|c_0| \leq \varepsilon$. Hence, by (2.2), $f$ is of the form

$$f(z) = u(z) + \frac{1}{\hat{h}(z)} \left\{ f(0) - u(0) + c_0 \int_0^z \hat{h}(\zeta) d\zeta \right\}$$

$$= \frac{c_0}{\lambda} + u(z) + \left( f(0) - u(0) - \frac{c_0}{\lambda} \right) e^{-\lambda z} \quad (z \in \mathbb{C}) \tag{2.10}$$

for all $z \in \mathbb{C}$. Put

$$f_0(z) = u(z) + \left( f(0) - u(0) - \frac{c_0}{\lambda} \right) e^{-\lambda z} \quad (z \in \mathbb{C}), \tag{2.11}$$

then $T_h f_0 = g$ and

$$|f(z) - f_0(z)| = \left| \frac{c_0}{\lambda} \right| \leq \frac{\varepsilon}{|\lambda|} \tag{2.12}$$

for every $z \in \mathbb{C}$ so that $T_h$ has the HUS with an HUS constant $1/|\lambda|$.

(b)$\Rightarrow$(a). Put

$$f_1(z) = \frac{1}{\hat{h}(z)} \int_0^z \hat{h}(\zeta) d\zeta \quad (z \in \mathbb{C}). \tag{2.13}$$

Then we obtain $T_h f_1 = 1$. Let $K < \infty$ be an HUS constant for $T_h$. Since $T_h$ has the HUS, there is an $f_2 \in H(\mathbb{C})$, such that $T_h f_2 = 0$ and

$$\sup_{z \in \mathbb{C}} |f_1(z) - f_2(z)| \leq K. \tag{2.14}$$

Note that $f_2$ is of the form $f_2(z) = f_2(0)/\hat{h}(z)$ for all $z \in \mathbb{C}$, since $T_h f_2 = 0$. Lemma 2.1, applied to $f_1$, yields that $h$ is a constant function. If $h$ were 0, then (2.13) would be written in the form $f_1(z) = z$ for $z \in \mathbb{C}$, and hence from (2.14), $\sup_{z \in \mathbb{C}} |z - f_2(0)| \leq K < \infty$, which is a contradiction. Thus, we conclude that $h$ is a nonzero constant function.

\[ \square \]

**Theorem 2.3.** Suppose $\lambda \in \mathbb{C} \setminus \{0\}$, $f, g \in H(\mathbb{C})$, and $\sup_{z \in \mathbb{C}} |T_\lambda f(z) - g(z)| < \infty$. Then there exists a unique $f_0 \in H(\mathbb{C})$ such that $T_\lambda f_0 = g$ and

$$\sup_{z \in \mathbb{C}} |f(z) - f_0(z)| < \infty. \tag{2.15}$$

Furthermore, $1/|\lambda|$ is the HUS constant for $T_\lambda$.

**Proof.** The existence of such a function $f_0 \in H(\mathbb{C})$ is proved by Theorem 2.2, and so we need to show only the uniqueness. Suppose $f_1 \in H(\mathbb{C})$ and $f_2 \in H(\mathbb{C})$ satisfy $T_\lambda f_j = g$ and

$$\sup_{z \in \mathbb{C}} |f(z) - f_j(z)| < \infty \quad (j = 1, 2) \tag{2.16}$$

for $j = 1, 2$. Since $T_\lambda f_j = g$,

$$f_j(z) = e^{-\lambda z} \left\{ f_j(0) + \int_0^z g(\zeta)e^{\lambda \zeta} d\zeta \right\} \quad (z \in \mathbb{C}) \tag{2.17}$$
for \( j = 1, 2 \), and hence
\[
f_1(z) - f_2(z) = (f_1(0) - f_2(0)) e^{-\lambda z} \quad \forall z \in \mathbb{C}.
\] (2.18)

It follows from (2.16) that \( f_1 - f_2 \) is constant by Liouville’s theorem. Therefore, \( f_1(0) = f_2(0) \) by (2.18), which implies that \( f_1 = f_2 \), proving the uniqueness.

We show that \( 1/|\lambda| \) is the HUS constant for \( T_\lambda \). Indeed, \( 1/|\lambda| \) is an HUS constant by (2.12). Conversely, let \( K \) be an arbitrary HUS constant for \( T_\lambda \), and put
\[
f_2(z) = \frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda z} \quad (z \in \mathbb{C}).
\] (2.19)

A simple calculation shows that \( f_2'(z) + \lambda f_2(z) = 1 \) for all \( z \in \mathbb{C} \), and hence \( \sup_{z \in \mathbb{C}} |T_\lambda f_2(z)| = 1 \). Then, there exists an \( f_3 \in H(\mathbb{C}) \) such that \( T_\lambda f_3 = 0 \) and \( \sup_{z \in \mathbb{C}} |f_2(z) - f_3(z)| = |\lambda| \) for \( z \in \mathbb{C} \), the uniqueness implies that \( f_3(z) = -\lambda^{-1} e^{-\lambda z} \), which proves \( 1/|\lambda| \leq K \). Thus, \( 1/|\lambda| \) is the HUS constant for \( T_\lambda \).

3. Stability for the Ger-type differential inequality. In this section, we consider the Ger-type stability problem. First, we show that the Ger-type stability does not hold in general. Indeed, the following proposition is true.

**Proposition 3.1.** For \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( \varepsilon > 0 \), there exists an \( f \in H(\mathbb{C}) \) with the following properties:

\[
\sup_{z \in \mathbb{C}} \left| \frac{f'(z)}{\lambda f(z)} - 1 \right| \leq \varepsilon,
\]
\[
\sup_{z \in \mathbb{C}} \left| \frac{f(z)}{c e^{\lambda z}} - 1 \right| = \sup_{z \in \mathbb{C}} \left| \frac{c e^{\lambda z}}{f(z)} - 1 \right| = \infty \quad \forall c \in \mathbb{C} \setminus \{0\}.
\] (3.1)

**Proof.** We associate to each \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( \varepsilon > 0 \) a function \( f \) defined by
\[
f(z) = e^{(\lambda + |\lambda| \varepsilon) z} \quad (z \in \mathbb{C}).
\] (3.2)

As above, we obtain
\[
f'(z) = (\lambda + |\lambda| \varepsilon) f(z) \quad (z \in \mathbb{C}),
\] (3.3)

so that
\[
\left| \frac{f'(z)}{\lambda f(z)} - 1 \right| = \varepsilon \quad \forall z \in \mathbb{C}.
\] (3.4)

If \( c \in \mathbb{C} \setminus \{0\} \), then we have
\[
\left| \frac{f(z)}{c e^{\lambda z}} - 1 \right| \geq \frac{1}{|c|} \left| e^{|\lambda| \varepsilon z} \right| - 1 \to \infty \quad (\text{Re} \, z \to \infty),
\]
\[
\left| c e^{\lambda z} \right| \left| \frac{f(z)}{f(z)} - 1 \right| \geq |c| \left| e^{-|\lambda| \varepsilon z} - 1 \right| \to \infty \quad (\text{Re} \, z \to -\infty).
\] (3.5)
and so
\[
\sup_{z \in \mathbb{C}} \left| \frac{f(z)}{ce^{\lambda z}} - 1 \right| = \sup_{z \in \mathbb{C}} \left| \frac{ce^{\lambda z}}{f(z)} - 1 \right| = \infty \quad \forall c \in \mathbb{C} \setminus \{0\}.
\]

One might ask when the Ger-type stability is true. We give an answer to this question. If the Ger-type stability holds, then the function \( f \in H(\mathbb{C}) \) must be of the form \( f(z) = f(0)e^{\lambda z} \). That is, the only solution to the differential equation \( y' = \lambda y \) has the Ger-type stability.

**Theorem 3.2.** Suppose \( \lambda \in \mathbb{C} \setminus \{0\} \), \( \varepsilon > 0 \), and \( f \in H(\mathbb{C}) \) satisfies \( f(z) \neq 0 \) for all \( z \in \mathbb{C} \) and (1.5) holds. Suppose
\[
\sup_{z \in \mathbb{C}} \left| \frac{f(z)}{ce^{\lambda z}} - 1 \right| \quad \text{or} \quad \sup_{z \in \mathbb{C}} \left| \frac{ce^{\lambda z}}{f(z)} - 1 \right|
\]
is finite for some \( c \in \mathbb{C} \setminus \{0\} \); then \( f \) is of the form \( f(z) = f(0)e^{\lambda z} \) for all \( z \in \mathbb{C} \).

**Proof.** It follows from (1.5) that \( 1 - f'/\lambda f \) is constant, say \( c_0 \in \mathbb{C} \), by Liouville’s theorem. Thus, \( f' = (1 - c_0)\lambda f \), and hence
\[
f(z) = f(0)e^{(1-c_0)\lambda z} \quad (z \in \mathbb{C}).
\]
Suppose that there is a \( c_1 \in \mathbb{C} \setminus \{0\} \) such that
\[
\sup_{z \in \mathbb{C}} \left| \frac{f(z)}{c_1 e^{\lambda z}} - 1 \right| < \infty.
\]
From (3.8), it follows that
\[
\sup_{z \in \mathbb{C}} \left| \frac{f(0)}{c_1} e^{-c_0 \lambda z} - 1 \right| < \infty,
\]
and hence \( c_0 \) must be 0, proving \( f(z) = f(0)e^{\lambda z} \) for all \( z \in \mathbb{C} \). Similarly, we can treat the case where
\[
\sup_{z \in \mathbb{C}} \left| \frac{c_2 e^{\lambda z}}{f(z)} - 1 \right| < \infty
\]
for some \( c_2 \in \mathbb{C} \setminus \{0\} \), and so the proof is omitted.

Thus far, we have treated entire functions. Finally, we consider the Ger-type stability problem in the category of holomorphic functions on a bounded region.

**Theorem 3.3.** Let \( 0 \in \Omega \) be a bounded convex region of \( \mathbb{C} \) and put \( M = \sup_{z \in \Omega} |z| \). Suppose \( \lambda \in \mathbb{C} \setminus \{0\} \), \( 0 \leq \varepsilon \leq 1 \), and \( f : \Omega \to \mathbb{C} \) is holomorphic such that \( f(z) \neq 0 \) for all \( z \in \Omega \) and
\[
\sup_{z \in \Omega} \left| \frac{f'(z)}{\lambda f(z)} - 1 \right| \leq \varepsilon.
\]
Then there are $K_\lambda > 0$ and $c \in \mathbb{C} \setminus \{0\}$ such that

$$\max \left\{ \sup_{z \in \Omega} \left| \frac{f(z)}{ce^{\lambda z}} - 1 \right|, \sup_{z \in \Omega} \left| \frac{ce^{\lambda z}}{f(z)} - 1 \right| \right\} \leq K_\lambda \varepsilon. \quad (3.13)$$

**Proof.** Put $g(z) = -1 + f'(z)/\lambda f(z)$ for $z \in \Omega$, and so

$$f'(z) = \lambda (1 + g(z)) f(z) \quad (z \in \Omega). \quad (3.14)$$

From (3.14), it follows that

$$f(z) = f(0)e^{\lambda z} \exp \int_0^z \lambda g(\zeta) d\zeta \quad (3.15)$$

for every $z \in \Omega$, and hence

$$\left| \frac{f(z)}{f(0)e^{\lambda z}} - 1 \right| = \left| \exp \int_0^z \lambda g(\zeta) d\zeta - 1 \right| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \left| \int_0^z \lambda g(\zeta) d\zeta \right|^n$$

$$\leq \sum_{n=1}^{\infty} \frac{|\lambda| e|\lambda|z^n}{n!} \leq (e^{|\lambda|M} - 1) \varepsilon \quad (3.16)$$

for all $z \in \Omega$. Similarly, we can show that

$$\sup_{z \in \Omega} \left| \frac{f(0)e^{\lambda z}}{f(z)} - 1 \right| \leq (e^{|\lambda|M} - 1) \varepsilon, \quad (3.17)$$

and so the proof is complete. \qed

**Acknowledgment.** The first and third authors are partially supported by the Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science.

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