FLAT SEMIMODULES

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To my dearest friend Najla Ali

We introduce and investigate flat semimodules and \( k \)-flat semimodules. We hope these concepts will have the same importance in semimodule theory as in the theory of rings and modules.

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1. Introduction. We introduce the notion of flat and \( k \)-flat. In Section 2, we study the structure ensuing from these notions. Proposition 2.4 asserts that \( V \) is flat if and only if \( (V \otimes_R -) \) preserves the exactness of all right-regular short exact sequences. Proposition 2.5 gives necessary and sufficient conditions for a projective semimodule to be \( k \)-flat. In Section 3, Proposition 3.3 gives the relation between flatness and injectivity. In Section 4, Proposition 4.1 characterizes the \( k \)-flat cancellable semimodules with the left ideals. Proposition 4.4 describes the relationship between the notions of projectivity and flatness for a certain restricted class of semirings and semimodules.

Throughout, \( R \) will denote a semiring with identity 1. All semimodules \( M \) will be left \( R \)-semimodules, except at cited places, and in all cases are unitary semimodules, that is, \( 1 \cdot m = m \) for all \( m \in M \) (\( m \cdot 1 = m \) for all \( m \in M \)) for all left \( R \)-semimodules \( _R M \) (resp., for all right \( R \)-semimodule \( _MR \)).

We recall here (cf. \([1, 2, 4, 7, 8]\)) the following facts.

(a) A semiring \( R \) is said to satisfy the left cancellation law if and only if for all \( a, b, c \in R \), \( a + b = a + c \Rightarrow b = c \). A semimodule \( M \) is said to satisfy the left cancellation law if for all \( m, m', m'' \in M \), \( m + m' = m + m'' \Rightarrow m' = m'' \).

(b) We say that a nonempty subset \( N \) of a left semimodule \( M \) is subtractive if and only if for all \( m, m' \in M \), \( m + m' \in N \) imply \( m' \in N \).

(c) A semiring \( R \) is called completely subtractive if \( _RR \) is a completely subtractive semimodule; and a left \( R \)-semimodule \( M \) is called completely subtractive if and only if for every subsemimodule \( N \) of \( M \), \( N \) is subtractive.

(d) A semimodule \( M \) is said to be free \( R \)-semimodule if \( M \) has a basis over \( R \).

(e) A semimodule \( C \) is said to be semicogenerated by \( U \) when there is a homomorphism \( \varphi : M \to \Pi A C \) such that \( \ker \varphi = 0 \). A semimodule \( C \) is said to be a semicogenerator when \( C \) semicgenerates every left \( R \)-semimodule \( M \).

(f) Let \( \alpha : M \to N \) be a homomorphism of semimodules. The subsemimodule \( \text{Im} \alpha \) of \( N \) is defined as follows: \( \text{Im} \alpha = \{ n \in N : n + \alpha(m') = \alpha(m) \text{ for some } m, m' \in M \} \). Also \( \alpha \) is
said to be a semimonomorphism if \( \ker \alpha = 0 \), to be a semi-isomorphism if \( \alpha \) is surjective and \( \text{Ker} \alpha = 0 \), to be an isomorphism if \( \alpha \) is injective and surjective, to be \( i \)-regular if \( \alpha(M) = \text{Im} \alpha \), to be \( k \)-regular if for all \( a, a' \in A \), \( \alpha(a) = \alpha(a') \) implying \( a + k = a' + k' \) for some \( k, k' \in \ker \alpha \), and to be regular if it is both \( i \)-regular and \( k \)-regular.

(g) An \( R \)-semimodule \( M \) is said to be \( k \)-regular if there exist a free \( R \)-semimodule \( F \) and a surjective \( R \)-homomorphism \( \alpha : F \to M \) such that \( \alpha \) is \( k \)-regular.

(h) The sequence \( K \xrightarrow{\alpha} M \xrightarrow{\beta} N \) is called an exact sequence if \( \text{Ker} \beta = \text{Im} \alpha \), and proper exact if \( \text{Ker} \beta = \alpha(K) \).

(i) A short sequence \( 0 \to K \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0 \) is said to be left \( k \)-regular right regular if \( \alpha \) is \( k \)-regular and \( \beta \) is right regular.

(j) For any two \( R \)-semimodules \( N, M \), \( \text{Hom}_R(N,M) := \{ \alpha : N \to M \mid \alpha \text{ is an } R \text{-homomorphism of semimodules} \} \) is a semigroup under addition. If \( M, N, \) and \( U \) are \( R \)-semimodules and \( \alpha : M \to N \) is a homomorphism, then \( \text{Hom}(\alpha, I_U) : \text{Hom}_R(N,U) \to \text{Hom}_R(M,U) \) is given by \( \text{Hom}(\alpha, I_U) : \gamma \mapsto \gamma \alpha \), where \( I_U \) is the identity on \( U \).

(k) If \( M \) is a right \( R \)-semimodule, \( N \) is a left \( R \)-semimodule, and \( T \) is an \( N \)-semimodule, then a function \( \theta : M \times \text{N} \to T \) is \( R \)-balanced if and only if, for all \( m, m' \in M \), for all \( n, n' \in N \), and for all \( r \in R \), we have

\[
\begin{align*}
(1) & \quad \theta(m + m', n) = \theta(m, n) + \theta(m', n), \\
(2) & \quad \theta(m, n + n') = \theta(m, n) + \theta(m, n'), \\
(3) & \quad \theta(mr, n) = \theta(m, rn).
\end{align*}
\]

Let \( R \) be a semiring, let \( M \) be a right \( R \)-semimodule, and let \( N \) be a left \( R \)-semimodule. Let \( A \) be the set \( M \times N \), and let \( U \) be the \( N \)-semimodule \( \oplus_A N \times \oplus_A N \). Let \( W \) be the subset of \( U \) consisting of all elements of the following forms:

\[
\begin{align*}
(1) & \quad (\alpha[m + m', n], \alpha[m, n] + \alpha[m', n]), \\
(2) & \quad (\alpha[m, n] + \alpha[m', n], \alpha[m + m', n]), \\
(3) & \quad (\alpha[m, n + n'], \alpha[m, n] + \alpha[m, n']), \\
(4) & \quad (\alpha[m, n] + \alpha[m', n'], \alpha[m, n + n']), \\
(5) & \quad (\alpha[mr, n], \alpha[m, rn]), \\
(6) & \quad (\alpha[m, rn], \alpha[mr, n]),
\end{align*}
\]

for \( m \) and \( m' \) in \( M \), \( n \) and \( n' \) in \( N \), and \( r \) in \( R \), and where \( \alpha[m, n] \) is the function from \( M \times N \) to \( N \) which sends \( (m, n) \) to 1 and sends every other element of \( M \times N \) to 0. Let \( U' \) be the \( N \)-subsemimodule of \( U \) generated by \( W \). Define \( N \) congruence relation \( \equiv \) on \( \oplus_A N \) by setting \( \alpha \equiv \alpha' \) if and only if there exists an element \( (\beta, \gamma) \in U' \) such that \( \alpha + \beta = \alpha' + \gamma \). The factor \( N \)-semimodule \( \oplus_A N / \equiv \) will be denoted by \( M \otimes_R N \), and is called the tensor product of \( M \) and \( N \) over \( R \).

(A) A left \( R \)-semimodule \( P \) is said to be projective semimodule if and only if for each surjective \( R \)-homomorphism \( \varphi : M \to N \), the induced homomorphism \( \overline{\varphi} : \text{Hom}_R(P, M) \to \text{Hom}_R(P, N) \) is surjective.

2. Flat and \( k \)-flat semimodules. In this section, we discuss the structure of flat and \( k \)-flat semimodules. \textbf{Proposition 2.4} asserts that \( V \) is flat if and only if \( (V \otimes_R -) \) preserves the exactness of all left \( k \)-regular right regular short sequences. In \textbf{Proposition 2.5}, we give the necessary and sufficient condition for the projective right semimodule to be \( k \)-flat relative to a cancellable left semimodule.
**Definition 2.1.** A semimodule $V_R$ is flat relative to a semimodule $RM$ (or that $V$ is $M$-flat) if and only if for every subsemimodule $K \leq M$, the sequence $0 \to V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M$ is proper exact (i.e., $\text{Ker}(I_V \otimes_R i_K) = 0$) where $I_V \otimes_R i_K(\nu = iK(k))$. A semimodule $V_R$ that is flat relative to every left $R$-semimodule is called a flat right $R$-semimodule.

**Definition 2.2.** A semimodule $V_R$ is $k$-flat relative to a semimodule $RM$ (or that $V$ is $Mk$-flat) if and only if for every subsemimodule $K \leq M$, the sequence $0 \to V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M$ is proper exact and $I_V \otimes i_K$ is $k$-regular (i.e., $I_V \otimes_R i_K$ is injective). A semimodule $V_R$ that is $k$-flat relative to every right $R$-semimodule is called a $k$-flat right $R$-semimodule. Thus, if $V_R$ is $k$-flat relative to $R$, then $V_R$ is flat relative to $R$.

Our next result shows that the class of flat and $k$-flat semimodules is closed under direct sums.

**Proposition 2.3.** Let $(V_\alpha)_{\alpha \in A}$ be an indexed set of right $R$-semimodules. Then $\oplus_{\alpha} V_\alpha$ is $M$-flat ($k$-flat) if and only if each $V_\alpha$ is $M$-flat ($k$-flat).

**Proof.** Let $M$ be a left $R$-semimodule and $K$ a subsemimodule of $M$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
V_\alpha \otimes K & \xrightarrow{I_{V_\alpha} \otimes i_K} & V_\alpha \otimes M \\
\pi_\alpha \downarrow & & \downarrow \pi_\alpha \\
\oplus (V_\alpha \otimes K) & \xrightarrow{\theta} & \oplus (V_\alpha \otimes M) \\
\phi \downarrow & & \phi' \\
(\oplus V_\alpha) \otimes K & \xrightarrow{I_{\oplus V_\alpha} \otimes i_K} & (\oplus V_\alpha) \otimes M,
\end{array}
\]

where $\pi_\alpha : \oplus (V_\alpha \otimes K) \to V_\alpha \otimes K$ and $i_\alpha : V_\alpha \otimes K \to \oplus (V_\alpha \otimes K)$ are defined respectively by $\pi_\alpha : (v_\alpha \otimes k_i)h v_\alpha \otimes k_\alpha$ and $i_\alpha : v_\alpha \otimes k_\alpha h (v_\alpha \otimes k_i)$, where $v_\alpha \otimes k_i = 0$ if $i \neq \alpha$ and $v_\alpha \otimes k_i = v_\alpha \otimes k_\alpha$ if $\alpha = i$; $\phi$ and $\phi'$ are the isomorphisms of [8, Proposition 5.4] given by $\phi[(v_\alpha) \otimes k] = (v_\alpha \otimes k)$ and $\theta(v_\alpha \otimes k) = (v_\alpha \otimes i(k))$. Now suppose that $\oplus V_\alpha$ is $M$-flat ($k$-flat). If $I_{V_\alpha} \otimes i_K(\nu' = iK(k')) = 0[I_{V_\alpha} \otimes i_K((v_\alpha \otimes k) = I_{V_\alpha} \otimes i_K((v_\alpha \otimes k'))]$, then by the above diagram we have $(v_\alpha) \otimes i_K(k) = 0[(v_\alpha) \otimes i(k) = (v_\alpha') \otimes i(k')]$. Since $\oplus V_\alpha$ is flat ($k$-flat), then $(v_\alpha) \otimes k = 0[(v_\alpha) \otimes k = (v_\alpha') \otimes k']$. Again by (2.1), $(v_\alpha) \otimes k = 0$ whence $v_\alpha \otimes k = 0[(v_\alpha) \otimes k = (v_\alpha') \otimes k']$, whence $v_\alpha \otimes k = (v_\alpha') \otimes k'$. Therefore $V_\alpha$ is flat ($k$-flat).

Conversely, suppose that $V_\alpha$ is $M$-flat ($k$-flat) for each $\alpha \in A$. If $I_{\oplus V_\alpha} \otimes i_K((v_\alpha) \otimes k) = 0[I_{\oplus V_\alpha} \otimes i_K((v_\alpha) \otimes k) = I_{\oplus V_\alpha} \otimes i_K((v_\alpha') \otimes k')]$, then by the above diagram we have $v_\alpha \otimes k = 0[v_\alpha \otimes i(k) = v_\alpha' \otimes i(k')]$ for each $\alpha \in A$. Since $V_\alpha$ is flat ($k$-flat), then $v_\alpha \otimes k = 0[v_\alpha \otimes k = v_\alpha' \otimes k']$ for each $\alpha$. Therefore, $(v_\alpha) \otimes k = 0[(v_\alpha) \otimes k = (v_\alpha') \otimes k']$. Again by (2.1), $(v_\alpha) \otimes k = 0[(v_\alpha) \otimes k = (v_\alpha') \otimes k']$. Thus $\oplus V_\alpha$ is flat ($k$-flat).

**Proposition 2.4.** Let $M$ be a left $R$-semimodule. A right $R$-semimodule $V$ is $M$-flat if and only if the functor $(V \otimes_R -)$ preserves the exactness of all left $k$-regular right regular
short exact sequences with middle term \( M \):

\[
0 \rightarrow R K \xrightarrow{\alpha} R M \xrightarrow{\beta} R N \rightarrow 0. \tag{2.2}
\]

**Proof.** “If” part. Let \( 0 \rightarrow R K \xrightarrow{\alpha} R M \xrightarrow{\beta} R N \rightarrow 0 \) be a left \( k \)-regular right regular exact sequence. Since \( V_R \) is \( R M \)-flat, then using [8, Theorem 5.5(2)], the sequence

\[
0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes \alpha} V \otimes_R M \xrightarrow{I_V \otimes \beta} V \otimes_R N \rightarrow 0 \tag{2.3}
\]

is exact.

“Only if” part. Let \( R K \leq R M \). Consider the following exact sequence:

\[
0 \rightarrow K \xrightarrow{i_K} M \xrightarrow{m_{\text{mi}}K} M/\text{Im} i_K \rightarrow 0. \tag{2.4}
\]

By hypothesis, \( 0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M \) is an exact sequence. Thus \( V \) is \( M \)-flat.

Our next result gives a necessary and sufficient condition for a projective semimodule to be \( k \)-flat relative to a cancellable semimodule \( M \).

**Proposition 2.5.** Let \( V_R \) be projective and \( R M \) cancellable. Then, \( V \) is \( Mk \)-flat if and only if the functor \( (V \otimes_R -) \) preserves the exactness of all left \( k \)-regular right regular short exact sequences

\[
0 \rightarrow R K \xrightarrow{\alpha} R M \xrightarrow{\beta} R N \rightarrow 0. \tag{2.5}
\]

**Proof.** “If” part. Let \( 0 \rightarrow R K \xrightarrow{\alpha} R M \xrightarrow{\beta} R N \rightarrow 0 \) be a left \( k \)-regular right regular exact sequence. Since \( V_R \) is \( R M \) \( k \)-flat, then \( V_R \) is \( R M \)-flat. By using Proposition 2.4, the sequence

\[
0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes \alpha} V \otimes_R M \xrightarrow{I_V \otimes \beta} V \otimes_R N \rightarrow 0 \tag{2.6}
\]

is exact.

“Only if” part. Let \( K \leq M \). Consider the following exact sequence:

\[
0 \rightarrow K \xrightarrow{i_K} M \xrightarrow{m_{\text{mi}}K} M/\text{Im} i_K \rightarrow 0. \tag{2.7}
\]

Since \( V \) is projective and \( M \) is cancellable, then by using [9, Proposition 1.16], \( I_V \otimes i_K \) is \( k \)-regular. By hypothesis, \( 0 \rightarrow V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M \) is an exact sequence. Thus \( V \) is \( Mk \)-flat.

3. **Flatness via injectivity.** We will discuss the relation between the injectivity and flatness. By \((\cdot)^*\) we mean the functor \( \text{Hom}_N(-, C) \), where \( C \) is a fixed injective semico-generator cancellative \( N \)-semimodule.

**Remark 3.1.** If \( U \) is a right \( R \)-semimodule, then \( U^* \) is a left \( R \)-semimodule.

**Proof.** Let \( \alpha \in \text{Hom}_R(U, C) \) and let \( r \in R \). Define \( r\alpha(u) = \alpha(ur) \). If \( s \in R \), then \( s(r\alpha)u = (r\alpha)(us) = \alpha(usr) = (sr)\alpha(u) \). Therefore, \( U^* \) is a left \( R \)-semimodule.

\( \square \)
We state and prove the following lemma, analogous to the one on modules which is needed in the proof of Proposition 3.3.

**Lemma 3.2.** Let $R$ be a semiring, let $M$ and $M'$ be left $R$-semimodules, and let $U$ be a right $R$-semimodule. Let $T$ be a cancellative $N$-semimodule. If $\alpha : M' \to M$ is an $R$-homomorphism, then there exist $N$-isomorphisms $\varphi$ and $\varphi'$ such that the following diagram commutes:

\[
\begin{array}{cccc}
\Hom_R(M, \Hom_N(U, T)) & \xrightarrow{\Hom_R(\alpha, \Hom_N(U, T))} & \Hom_R(M', \Hom_N(U, T)) \\
\varphi & & \varphi' \\
\Hom_N((U \otimes_R M), T) & \xrightarrow{\Hom_N((l_U \otimes \alpha), I_T)} & \Hom_N((U \otimes_R M'), T).
\end{array}
\] (3.1)

**Proof.** By [7, Proposition 14.15], there exists an $N$-isomorphism $\varphi : \Hom_R(M, \Hom_N(U, T)) \to \Hom_R(M \otimes U, T)$ (3.2) given by $\varphi(y) : u \otimes mh(y(m))u$. Then with a parallel definition for $\varphi'$, we have

\[
\varphi' h \Hom_R(\alpha, l_{\Hom_N(U, T)})(y)(u \otimes m') = \varphi'(y \alpha)(u \otimes m') = (y \alpha)(m')(u) = y(\alpha(m'))(u) = \varphi(y)(u \otimes \alpha(m'))
\]

\[
= \varphi(y) h (l_U \otimes \alpha)(u \otimes m') = \Hom_N(l_U \otimes \alpha, I_T)(\varphi(y))(u \otimes m'),
\] (3.3)

and the diagram commutes. \hfill \qed

**Proposition 3.3.** Let $M$ be a left $R$-semimodule.

1. If the right $R$-semimodule $V$ is $M_k$-flat, then $V^*$ is $M$-injective.
2. If $V^*$ is $M$-injective, then $V$ is $M$-flat.

**Proof.** (1) Let $K$ be a subsemimodule of $M$. Since $V$ is $M_k$-flat, then the sequence $0 \to V \otimes K \xrightarrow{i_V \otimes i_K} V \otimes M$ is proper exact, and $I_V \otimes i_K$ is $k$-regular. By Lemma 3.2, we have the following commutative diagram:

\[
\begin{array}{cccc}
\Hom_R(M, V^*) & \xrightarrow{\Hom_R(i_K, I_V^*)} & \Hom_R(K, V^*) & \to 0 \\
\varphi' \downarrow & & \varphi \downarrow \\
(V \otimes M)^* & \xrightarrow{\Hom(I_V \otimes i_K, I_C)} & (V \times K)^* & \to 0,
\end{array}
\] (3.4)

where $\varphi'$ and $\varphi$ are $N$-isomorphisms. It follows that the top row is proper exact if and only if the bottom row is proper exact, whence by [6, Proposition 3.1], $V^*$ is injective.

(2) If $V^*$ is injective, then

\[
\Hom(M, V^*) \xrightarrow{\Hom(i_K, I_V^*)} \Hom(K, V^*) \to 0
\] (3.5)
is proper exact. Again by the above diagram,

\[(V \otimes M)^* \xrightarrow{\text{Hom}(I_V \otimes i_K, I_e)} (V \otimes K)^* \rightarrow 0 \quad (3.6)\]

is proper exact. Hence, the sequence is exact. Since \(C\) is a semicogenerator, then by [3, Proposition 4.1], the sequence \(0 \rightarrow V \otimes K \rightarrow V \otimes M\) is an exact sequence. Hence, \(V\) is \(M\)-flat.

4. Cancellable semimodules. In this section, we deal with cancellable semimodules. We characterize \(k\)-flat cancellable semimodules by means of left ideals.

**Proposition 4.1.** The following statements about a cancellable right \(R\)-semimodule \(V\) are equivalent:

1. \(V\) is \(k\)-flat relative to \(RR\);
2. for each (finitely generated) left ideal \(I \leq RR\), the surjective \(N\)-homomorphism \(\varphi : V \otimes R I \rightarrow VI\) with \(\varphi(v \otimes a) = va\) is a \(k\)-regular semimonomorphism.

**Proof.** (1)\(\Rightarrow\)(2). Since \(V\) is cancellable, then by using [7, Proposition 14.16], \(V \otimes R R \cong V\). Consider the following commutative diagram:

\[
\begin{array}{ccc}
V \otimes R I & \xrightarrow{i_V \otimes i_I} & V \otimes R R \\
\varphi \downarrow & & \theta \downarrow \\
VI & \xrightarrow{i_V I} & V,
\end{array}
\]

where \(\theta\) is the isomorphism of [7, Proposition 14.16]. Since \(\varphi : V \times I \rightarrow VI\) given by \(\varphi(v, i) = vi\) is an \(R\)-balanced function, then by using [7, Proposition 14.14], there is an exact unique \(N\)-homomorphism \(\varphi : V \otimes I \rightarrow V\) satisfying the condition \(\varphi(v \otimes i) = \varphi(v, i)\). Since \(V\) is \(k\)-flat relative to \(RR\), then \(\varphi(\Sigma v_i \otimes a_i) = \varphi(\Sigma v'_i \otimes a'_i)\), then \(\varphi(\Sigma v_1 \otimes a_1) = \varphi(\Sigma v'_1 \otimes a'_1)\).

(2)\(\Rightarrow\)(1). Again consider the above diagram. Let \(I\) be any left ideal of \(R\) and let \(i_I : (\Sigma v_1 \otimes a_1) = I_V \otimes R_1 I(\Sigma v'_1 \otimes a'_1)\), where \(\Sigma v'_1 \otimes a'_1, \Sigma v_1 \otimes a_1 \in V \otimes R I\). Let \(K_1 = \Sigma Ra_i, K_2 = \Sigma Ra'_i,\) and \(K = K_1 + K_2\). Now \(\varphi(\Sigma v_i \otimes a_i) = \varphi(\Sigma v'_i \otimes a'_i)\), whence \(\Sigma v_i a_i = \Sigma v'_i a'_i\). Now consider the following diagram, where \(i_K : K \rightarrow I\) is the inclusion map:

\[
\begin{array}{ccc}
V \otimes K & \xrightarrow{i_V \otimes i_K} & V \\
\varphi_K \downarrow & & \varphi \downarrow \\
VK & \xrightarrow{i_V K} & V.
\end{array}
\]

By hypothesis, \(\varphi_K\) is monic. Thus, \(\Sigma i v_i \otimes a_i = \Sigma i v'_i \otimes a'_i\) as an element of \(V \otimes K\). Hence, \(I_V \otimes R K(\Sigma v_i \otimes a_i) = I_V \otimes K(\Sigma v'_i \otimes a'_i) \in V \otimes I\), and \(\Sigma v_i \otimes a_i = \Sigma v'_i \otimes a'_i\) as an element of \(V \otimes I\). Therefore, \(I_V \otimes R i_I\) is monic. Hence, \(V\) is \(k\)-flat relative to \(RR\).
Proposition 4.2. Let $M$ be a cancellable left $R$-semimodule. Then $R_R$ is $M_k$-flat.

Proof. Let $i_K : K \to M$ be the inclusion homomorphism. By [7, Proposition 14.16], $R \otimes_R K \simeq K$ and $R \otimes R M \simeq M$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
R \otimes_R K & \xrightarrow{i_R \otimes i} & R \otimes_R M \\
\downarrow{=} & & \downarrow{=} \\
K & \xrightarrow{i_K} & M,
\end{array}
\]

(4.3)

since $i_K$ is injective, then $I \otimes_R i_K$ is injective. □

Corollary 4.3. Let $M$ be a cancellable left $R$-semimodule. Then every free $R$-semimodule is $M_k$-flat.

Proof. The proof is immediate from Propositions 2.3 and 4.2. □

In module theory every projective module is flat. Now we see that this is true for certain special semimodules.

Proposition 4.4. Let $M$ be a cancellable left $R$-semimodule, where $R$ is a cancellative completely subtractive semiring. Then every $k$-regular projective $R$-semimodule $P$ is $M_k$-flat.

Proof. By using [5, Theorem 19], $P$ is isomorphic to a direct summand of a free semimodule $F$. By Corollary 4.3, $F$ is $M_k$-flat. Hence, by using Proposition 2.3, $P$ is $M_k$-flat. □

Corollary 4.5. Let $M$ be a $k$-regular left $R$-semimodule and $R$ a cancellative completely subtractive semiring. Then every $k$-regular projective $R$-semimodule $P$ is $M_k$-flat.

Proof. We only need to show that $M$ is cancellable. Since $M$ is $k$-regular, then there exists a free $R$-semimodule $F$ such that $\varphi : F \to M$ is surjective. Let $m_1 + m = m_2 + m$, where $m_1, m_2, m \in M$. Since $\varphi$ is surjective, then $\varphi(a_1) + \varphi(a) = \varphi(a_2) + \varphi(a)$, where $\varphi(a_1) = m_1$, $\varphi(a) = m$, and $\varphi(a_2) = m_2$. Since $\varphi$ is $k$-regular, then $a_1 + a + k_1 = a_2 + a + k_2$, where $k_1, k_2 \in \text{Ker} \varphi$. Since $F$ is cancellable, then $a_1 + k_1 = a_2 + k_2$. Hence $\varphi(a_1) = \varphi(a_2)$. □

Proposition 4.6. Let $M$ be a cancellable left $R$-semimodule. If $V$ is a free $R$-semimodule, then the following assertions hold:

(a) $V$ is $M_k$-flat;
(b) $V^*$ is $M$-injective.

Proof. By using Corollary 4.3, $V$ is $M_k$-flat.

(i)⇒(ii). The proof is immediate from Proposition 3.3. □

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