We generalize min-neighborhood groups to arbitrary $T$-neighborhood groups, where $T$ is a continuous triangular norm. In particular, we point out that our results accommodate the previous theory on min-neighborhood groups due to T. M. G. Ahsanullah. We show that every $T$-neighborhood group is $T$-uniformizable, therefore, $T$-completely regular. We also present several results of $T$-neighborhood groups in conjunction with $TI$-groups due to J. N. Mordeson.

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1. Introduction. Menger in [19] introduces an important class of $T$-uniformities ($T$ being a continuous $t$-norm) that is generated by a probabilistic metric [21]. Motivated by the Menger’s $T$-uniformities, Höhle [13] brought into light in his celebrated article the idea of probabilistic metrization of fuzzy uniformities. While developing his theory, he showed that a fuzzy $T$-uniformity is probabilistic metrizable if and only if it is Hausdorff-separated and has a countable base. He also pointed out that when $T = \text{min}$ is considered, his fuzzy $T$-uniformity reduces to min-fuzzy uniformity of R. Lowen—a fuzzy uniformity widely used over the years. One of the interesting features of this min-fuzzy uniformity [16] is that it gives rise to a fuzzy neighborhood space [17]; an interesting and very well-behaved class of fuzzy topological spaces [15], used by many authors in a wide variety of ways. Among the prominent classes of so called fuzzy neighborhood spaces are, for instance, Katsaras linear fuzzy neighborhood spaces [14], fuzzy metric neighborhood spaces [18], fuzzy neighborhood groups, rings, modules, algebras, and commutative division rings [1, 2, 3, 4, 5].

Very recently, following the famous articles of Menger [19], Höhle [13], Frank [9], Hashem and Morsi [10, 11, 12] introduced a class of fuzzy topological spaces [15] as they put it: fuzzy $T$-neighborhood spaces herein called $T$-neighborhood spaces—a natural generalization of min-fuzzy neighborhood spaces of Lowen [17]. Our main target here in this article is to generalize the notion of min-fuzzy neighborhood groups introduced in [2]. We show that every $T$-neighborhood group is a $T$-uniform space, and therefore, a $T$-complete regular space in the sense of Hashem and Morsi [12]. We also generalize the two important characterization theorems, which give necessary and sufficient condition for a $T$-neighborhood system and a group structure to be compatible, and a prefilter to be a $T$-neighborhood prefilter.

As an application, we present some results on $T$-neighborhood groups in conjunction with Mordeson’s $TI$-groups [7], which we believe will open the opportunities to look into further work on fuzzy algebraic structures in connection with the $T$-neighborhood groups.
2. Preliminaries. Let $T$ be a continuous two-place function (known as continuous triangular norm or $t$-norm) mapping from the closed unit square to the closed unit interval satisfying certain conditions. In other words, $T : I \times I \to I$, $(\alpha, \beta) \mapsto \alpha T \beta$, satisfying the following conditions:

(Ta) $0T0 = 0, \alpha T 1 = \alpha$ for all $\alpha \in I$;
(Tb) $\alpha T \beta = \beta T \alpha$ for all $(\alpha, \beta) \in I \times I$;
(Tc) if $\alpha \leq \beta$ and $\gamma \leq \delta$, then $\alpha T \gamma \leq \beta T \delta$ for all $\alpha, \beta, \gamma, \delta \in I$;
(Td) $(\alpha T \beta) T \gamma = \alpha T (\beta T \gamma)$ for all $\alpha, \beta, \gamma \in I$.

**Definition 2.1** [6, 16]. A nonempty subset $\mathcal{B} \subset I^G$ is called a prefilterbase if and only if the following conditions are true:

(PB1) $0 \notin \mathcal{B}$;
(PB2) for all $\nu_1, \nu_2 \in \mathcal{B}$, there exists $\nu \in \mathcal{B}$ such that $\nu \leq \nu_1 \wedge \nu_2$.

If $\mathcal{B}$ is a prefilterbase in $I^G$, then by its saturation we understand the following collection:

$$\mathcal{B}^- = \{ \nu : G \to I ; \forall \epsilon > 0 \exists \nu_\epsilon \in \mathcal{B} \ni \nu_\epsilon - \epsilon \leq \nu \}.$$ (2.1)

**Definition 2.2** [10, 11, 12]. A $T$-neighborhood space is an $I$-topological space [15] $(G, -)$ whose closure operator “$-$” is induced by some indexed family $\Omega = (\Omega(x))_{x \in G}$ of prefilterbases in $I^G$ defined by

$$\tilde{\xi}(x) = \inf_{\nu \in \Omega(x)} \sup_{z \in G} \nu(z) T \nu(z), \quad \forall \xi \in I^G, \ x \in G.$$ (2.2)

**Theorem 2.3** [10, 11, 12]. A family $\Omega = (\Omega(x))_{x \in G}$ of prefilterbases in $I^G$ is a $T$-neighborhood base in $G$ if and only if it satisfies the following two properties for all $x \in G$:

(TB1) for all $\nu \in \Omega(x)$, $\nu(x) = 1$;
(TB2) for all $\nu \in \Omega(x)$, there exists a family $(\nu_{y, \epsilon} \in \Omega(y))_{(y, \epsilon) \in G \times I_0}$ which satisfies for all $(y, \epsilon) \in G \times I_0$,

$$\sup_{z \in G} [\nu_{x, \epsilon}(z) T \nu_{z, \epsilon}(y)] \leq \nu(y) + \epsilon.$$ (2.3)

The family $\Omega$ is said to be a $T$-neighborhood basis for $(G, -)$, and every $\nu \in \Omega(x)$ is called $T$-neighborhood at $x$. This $I$-topology is denoted by $t^T(\Omega)$. However, from now on we will be calling the triple $(G, - , t(\Omega))$ the $T$-neighborhood space with $\Omega$ a $T$-neighborhood base in $G$.

**Theorem 2.4** [10, 11, 12]. Let $(G_1, -, t(\Omega_1))$ and $(G_2, -, t(\Omega_2))$ be $T$-neighborhood spaces with $T$-neighborhood bases $\Omega_1$ and $\Omega_2$, respectively. Then a function $f : G_1 \to G_2$ is continuous at $x \in G_1$ if and only if

$$\forall \mu \in \Omega_2(f(x)),$$ $f^{-1}(\mu) \in \Omega_1(x)^-,$

$$\forall \mu_2 \in \Omega_2(f(x)) \forall \epsilon > 0 \exists \mu_1 \in \Omega_1(x) \ni \mu_1 - \epsilon \leq f^{-1}(\mu_2),$$

$$\lfloor f^{-1}(\sigma) \rfloor (x) \leq \lfloor f^{-1}(\tilde{\sigma}) \rfloor (x), \quad \forall \sigma \in I^{G_2}.$$

(2.4)
If $\Lambda, \Gamma \in I^{G \times G}$ and $\nu \in I^G$, then $T$-section of $\Lambda$ over $\nu$ is given by

$$\Lambda(\nu)_T(x) = \sup_{y \in G} \nu(y)T\Lambda(y, x) \quad \forall x \in G. \quad (2.5)$$

The $T$-composition of $\Lambda$ and $\Gamma$ is defined as

$$\Lambda \circ_T \Gamma(x, y) = \sup_{z \in G} [\Gamma(x, z)T\Lambda(z, y)] \quad \forall (x, y) \in G \times G. \quad (2.6)$$

$\Gamma$ is called symmetric if $\Gamma^T = \Gamma$, that is, $\Gamma(y, x) = \Gamma(x, y)$, for all $(x, y) \in G \times G$.

**Definition 2.5** [10, 11, 12]. A subset $\mathcal{B} \subset I^{G \times G}$ is called a $T$-uniform base on a set $G$ if and only if the following properties are fulfilled:

(TUB1) $\mathcal{B}$ is a prefilterbase;
(TUB2) for all $x \in G$, for all $\nu \in \mathcal{B}$, $\nu(x, x) = 1$;
(TUB3) for all $\beta \in \mathcal{B}$, for all $\epsilon > 0$, there exists $\beta_\epsilon \in \mathcal{B}$ such that $\beta_\epsilon \circ_T \beta_\epsilon - \epsilon \leq \beta$;
(TUB4) for all $\beta \in \mathcal{B}$, for all $\epsilon > 0$, there exists $\beta_\epsilon \in \mathcal{B}$ such that $\beta_\epsilon - \epsilon \leq \beta$.

The collection $\mathcal{B}$ of fuzzy subsets of $G \times G$ is called $T$-quasi-uniform base on a set $G$ if and only if it fulfills the preceding conditions (TUB1), (TUB2), and (TUB3), while $\mathcal{U}$ is called $T$-quasi-uniformity if and only if $\mathcal{B} = \mathcal{U}$. A $T$-uniformity $\mathcal{U}$ is a saturated $T$-uniform base $\mathcal{B}$.

**Theorem 2.6** [10, 11, 12]. If $\mathcal{B}$ is a $T$-quasi-uniform base on a set $G$, then for all $x \in G$, the family

$$\Sigma(x) = \{\beta(1_x) \mid \beta \in \mathcal{B}^{-}\} = \{\beta(1_x) \mid \beta \in \mathcal{B}^{-}\} \quad (2.7)$$

is a $T$-neighborhood system on $G$.

**Proposition 2.7** [10, 11, 12]. Let $(G, \mathcal{U})$ be a $T$-quasi-uniform space. Then the closure of the $T$-neighborhood space $(G, t(\mathcal{U}))$ is given by

$$\tilde{\mu} = \inf_{\sigma \in \mathcal{U}} \sigma(\mu)_T \quad \forall \mu \in I^G. \quad (2.8)$$

**Theorem 2.8** [10, 11, 12]. If $(G, \tau)$ is a topological space and $\mathcal{V}_\tau = (\mathcal{V}_\tau(x))_{x \in G}$ is its associated neighborhood system in $G$, then $(G, \tau, t(\mathcal{U}))$, a generated topological space, generated by $\tau$, is a $T$-neighborhood space with a $T$-neighborhood basis $\Omega = (\Omega(x))_{x \in G}$, where for all $x \in G$,

$$\begin{align*}
\Omega_1 := \Omega_1(x) &= \{1_M : G \to I; M \in \mathcal{V}_\tau(x)\} \subset I^G; \\
\Omega_2 := \Omega_2(x) &= \{1_M : G \to I; x \in M \in \tau\} \subset I^G; \\
\Omega_3 := \Omega_3(x) &= \{\nu : G \to I; \nu \text{ is l.s.c. in } x \text{ and } \nu(x) = 1\} \subset I^G.
\end{align*} \quad (2.9)
$$

Just for the sake of convenience, we provide the proof of the following proposition.

**Proposition 2.9.** A function $f : (G, \mathcal{V}_\tau) \to (G', \mathcal{V}'_{\tau'})$ between two topological spaces is continuous at a point $x \in G$ if and only if $f : (G, t(\Omega_\tau)) \to (G', t(\Omega'_{\tau'}))$ is continuous at $x \in G$ between two generated $T$-neighborhood spaces.
**Proof.** Let \( f : (G, \mathcal{V}_\tau) \to (G', \mathcal{V}'_{\tau'}) \) be continuous at \( x \in G \) and \( \mu' \in \Omega'_{\tau'}(f(x)) \); in view of Theorem 2.4, we show that \( f^{-1}(\mu') \in \Omega_{\tau}(x) \).

Choose \( M' \in \mathcal{V}'_{\tau'}(f(x)) \) such that \( \mu' = 1_{M'} \). This implies that there exists an \( M \in \mathcal{V}_{\tau}(x) \) such that \( f(M) \subset M' \), and hence for all \( \epsilon > 0 \),

\[
1_M(x) - \epsilon = 1 - \epsilon \leq 1_{f^{-1}(M')}(x) = f^{-1}(\mu').
\]  

(2.10)

With \( \mu = 1_M \), one obtains \( \mu - \epsilon \leq f^{-1}(\mu') \) implying that \( f^{-1}(\mu') \in \Omega_{\tau}(x) \).

Conversely, we show that the function \( f : (G, \mathcal{V}_{\tau}) \to (G', \mathcal{V}'_{\tau'}) \) is continuous at \( x \in G \). If \( U \in \mathcal{V}'_{\tau'}(f(x)) \), then \( 1_U \in \Omega'_{\tau'}(f(x)) \) implies \( f^{-1}(1_U) \in \Omega_{\tau}(x) \) by continuity of \( f \) between the generated spaces.

Thus, for all \( \epsilon > 0 \), there is a \( \mu = \mu_\epsilon \in \Omega_{\tau}(x) \) such that

\[
\mu - \epsilon \leq f^{-1}(1_U).
\]

(2.11)

This implies that for all \( \epsilon > 0 \), there exists a \( V_\epsilon \in \mathcal{V}_{\tau}(x) \) such that \( 1_{V_\epsilon} = \mu = \mu_\epsilon \) and

\[
1_{V_\epsilon} - \epsilon \leq 1_{f^{-1}(U)}.
\]

(2.12)

Now \( V_\epsilon \in \mathcal{V}_{\tau}(x) \) implies \( x \in V_\epsilon \) if and only if \( 1_{V_\epsilon}(x) = 1 \). Therefore,

\[
0 < 1 - \epsilon = 1_{V_\epsilon}(x) - \epsilon \leq 1_{f^{-1}(U)}(x) \Rightarrow 1_{f^{-1}(U)}(x) > 0 \Rightarrow 1_{f^{-1}(U)}(x)
= 1 \iff x \in f^{-1}(U).
\]

(2.13)

This means that \( V_\epsilon \subseteq f^{-1}(U) \) implies \( f^{-1}(U) \in \mathcal{V}_{\tau}(x) \). That is, \( f \) is continuous at \( x \in G \).

\[ \square \]

**Theorem 2.10** [11]. Let \((G, \tau)\) be an I-topological space. Then \((G, \tau)\) is a T-neighborhood space if and only if \( \alpha T\bar{\mu} = \alpha T\bar{\mu} \) for all \( \mu \in I^G \) and for all \( \alpha \in I \).

**Definition 2.11** [12]. An I-topological space \((X, \tau)\) is called \( T \)-completely regular if \( \tau \) is the initial \( I \)-topology for the family of all continuous functions from \((X, \tau)\) to \((\mathbb{D}^+, t^I(\mathcal{F}_\mathcal{X}))\).

Here, \( \mathbb{D}^+ \) stands for the collection of all distance distribution functions from \( \mathbb{R}^+ \) to the unit interval \( I \), and the pair \((\mathbb{D}^+, t^I(\mathcal{F}_\mathcal{X}))\) is the \( T \)-neighborhood space induced by the well-known Höhle’s probabilistic \( T \)-metric \( \mathcal{F}_\mathcal{X} \). For details, we refer to [12, 13].

**Theorem 2.12** [12]. \( T \)-complete regularity is equivalent to \( T \)-uniformizability.

### 3. Some results on \( T \)-neighborhood spaces

**Theorem 3.1.** Let \((G_1, \tau_1(\Omega_1))\) and \((G_2, \tau_2(\Omega_2))\) be two \( T \)-neighborhood spaces with bases \( \Omega_1 = (\Omega_1(x))_{x \in G_1} \) and \( \Omega_2 = (\Omega_2(x))_{x \in G_2} \) in \( G_1 \) and \( G_2 \), respectively. Then their \( T \)-product \((G_1 \times G_2, -\tau_1 \otimes, \tau(\Omega_1) \otimes_\tau \Omega_2)\) is the \( T \)-neighborhood space with base \( \Omega = \Omega_1 \otimes_\tau \Omega_2 \) defined by

\[
\Omega(x, y) = \{ v_1 \otimes_\tau v_2 \mid v_1 \in \Omega_1(x), \, v_2 \in \Omega_2(y) \},
\]

(3.1)
where \( \nu_1 \otimes_T \nu_2 \) is given as
\[
\nu_1 \otimes_T \nu_2 (x,y) = \nu_1(x)T\nu_2(y) \quad \forall (x,y) \in G_1 \times G_2.
\] (3.2)

Moreover,
\[
\nu_1 \otimes_T \nu_2 = \nu_1 \otimes_T \nu_2 \quad \forall \nu_1 \in I^{G_1}, \nu_2 \in I^{G_2}.
\] (3.3)

Conversely, if
\[
\nu_1 \otimes_T \nu_2 = \nu_1 \otimes_T \nu_2 \quad \forall \nu_1 \in I^{G_1}, \nu_2 \in I^{G_2},
\] (3.4)

then both the \( I \)-topological spaces \((G_1, \cdot)\) and \((G_2, \cdot)\) are \( T \)-neighborhood spaces.

**Proof.** First we show that for all \((x,y) \in G_1 \times G_2, \Omega(x,y)\) is a prefilterbase.

(PB1) Obviously, \( \Omega \neq \emptyset \) and \( \emptyset \notin \Omega \).

(PB2) Let \( \xi_1, \xi_2 \in \Omega(x,y) \), then there are \( \nu_1, \nu_2 \in \Omega_1(x) \) and \( \mu_1, \mu_2 \in \Omega_2(y) \) such that
\[\xi_1 = \nu_1 \otimes_T \mu_1 \text{ and } \xi_2 = \nu_2 \otimes_T \mu_2.\]

Now, \( \xi_1 \wedge \xi_2 = (\nu_1 \otimes_T \mu_1) \wedge (\nu_2 \otimes_T \mu_2). \) For any \((x,y) \in G_1 \times G_2, \)
\[
\xi_1(x,y) \wedge \xi_2(x,y) = (\nu_1 \otimes_T \mu_1)(x,y) \wedge (\nu_2 \otimes_T \mu_2)(x,y)
= (\nu_1(x)T\mu_1(y)) \wedge (\nu_2(x)T\mu_2(y))
\geq (\nu_1(x) \wedge \nu_2(x))T(\nu_1(y) \wedge \mu_2(y))
= (\nu_1 \wedge \nu_2)(x)T(\mu_1 \wedge \mu_2)(y).
\] (3.5)

Therefore, \( \xi_1 \wedge \xi_2(x,y) \geq \nu(x) \wedge \mu(y) \) for some \( \nu \in \Omega_1(x) \) and \( \mu \in \Omega_2(y) \), since both \( \Omega_1(x) \) and \( \Omega_2(y) \) are prefilterbases in \( G_1 \) and \( G_2 \), respectively.

This implies that \( \xi_1 \wedge \xi_2(x,y) \geq \nu \otimes_T \mu(x,y) = \xi(x,y) \) and \( \xi \in \Omega(x,y) \), and hence \( \xi \leq \xi_1 \wedge \xi_2 \), proving that \( \Omega(x,y) \) is a prefilterbase in \( G_1 \times G_2 \).

Now we prove the conditions of Theorem 2.3.

(TB1) If \( x \in G \) and \( \xi \in \Omega(x,x) \), then for some \( \nu \in \Omega_1(x) \) and \( \mu \in \Omega_2(x) \), we have
\[\xi(x,x) = \nu \otimes_T \mu(x,x), \quad \nu(x)T\mu(x) = T1 = 1.\]

(TB2) Let \( (x,y) \in G_1 \times G_2, \xi \in \Omega(x,y), \) and \( \epsilon \in I_0. \) Then there exists \( \nu \in \Omega_1(x) \) and \( \mu \in \Omega_2(y) \) such that \( \xi = \nu \otimes_T \mu. \)

Consequently, there is a family \((\nu_{y_1\epsilon} \in \Omega_1(y_1))_{(y_1,\epsilon) \in G_1 \times I_0},\) a \( T \)-kernel for \( \nu \) which satisfies for all \((y_1, \epsilon) \in G_1 \times I_0,\)
\[
\sup_{z_1 \in G_1} [\nu_{x,\epsilon}(z_1)T\nu_{z_1,\epsilon}(y_1)] \leq \nu(y_1) + \epsilon.
\] (3.6)

Also, there is a family \((\mu_{y_2\epsilon} \in \Omega_2(y_2))_{(y_2,\epsilon) \in G_2 \times I_0},\) a \( T \)-kernel for \( \mu \) which satisfies for all \((y_2, \epsilon) \in G_2 \times I_0,\)
\[
\sup_{z_2 \in G_2} [\mu_{x,\epsilon}(z_2)T\mu_{z_2,\epsilon}(y_2)] \leq \mu(y_2) + \epsilon.
\] (3.7)
Now for all \((y_1, y_2) \in G_1 \times G_2\),
\[
(\nu \otimes T \mu)(y_1, y_2) + \delta = [\nu(y_1) T \mu(y_2)] + \delta \geq (\nu(y_1) + \epsilon) T (\mu(y_2) + \epsilon)
\] (3.8)
with \(\epsilon = \epsilon_{T, \delta} > 0\).

This yields that
\[
(\nu \otimes T \mu)(y_1, y_2) + \delta \\
\geq \sup_{z_1 \in G_1} \left[ \nu_{x, \epsilon}(z_1) T \nu_{z_1, \epsilon}(y_1) \right] T \sup_{z_2 \in G_2} \left[ \mu_{y, \epsilon}(z_2) T \mu_{z_2, \epsilon}(y_2) \right] \\
= \sup_{(z_1, z_2) \in G_1 \times G_2} \left[ \left( \nu_{x, \epsilon}(z_1) T \mu_{y, \epsilon}(z_2) \right) T \left( \nu_{z_1, \epsilon}(y_1) T \mu_{z_2, \epsilon}(y_2) \right) \right] \\
= \sup_{(z_1, z_2) \in G_1 \times G_2} \left[ \left( \nu_{x, \epsilon} \otimes T \mu_{y, \epsilon}(z_1, z_2) \right) T \left( \nu_{z_1, \epsilon} \otimes T \mu_{z_2, \epsilon}(y_1, y_2) \right) \right] \\
\Rightarrow \sup_{(z_1, z_2) \in G_1 \times G_2} \left[ \nu_{x, \epsilon} \otimes T \mu_{y, \epsilon}(z_1, z_2) T \nu_{z_1, \epsilon} \otimes T \mu_{z_2, \epsilon}(y_1, y_2) \right] \\
\leq \nu \otimes T \mu(y_1, y_2) + \delta.
\] (3.9)

In order to prove the final part, we proceed as follows. Let \(\nu_1 \in I^{G_1}\), \(\nu_2 \in I^{G_2}\), and \((x, y) \in G_1 \times G_2\).

Then in view of Definition 2.2, we have
\[
\check{\nu}_1 \otimes \check{\nu}_2(x, y) = \inf_{\xi_1 \in \Omega_1(x)} \inf_{\xi_2 \in \Omega_2(y)} \sup_{z_1 \in G_1} \sup_{z_2 \in G_2} \left( \nu_1 \otimes T \nu_2 \right)(z_1, z_2) T \left( \xi_1 \otimes T \xi_2 \right)(z_1, z_2) \\
= \inf_{\xi_1 \in \Omega_1(x)} \inf_{\xi_2 \in \Omega_2(y)} \sup_{z_1 \in G_1} \sup_{z_2 \in G_2} \left( \nu_1(z_1) T \nu_2(z_2) \right) T \left( \xi_1(z_1) T \xi_2(z_2) \right) \\
= \inf_{\xi_1 \in \Omega_1(x)} \sup_{\xi_2 \in \Omega_2(y)} \sup_{z_1 \in G_1} \left( \nu_1(z_1) T \xi_1(z_1) T \nu_2(z_2) T \xi_2(z_2) \right) \\
= \inf_{\xi_1 \in \Omega_1(x)} \sup_{z_1 \in G_1} \nu_1(z_1) T \xi_1(z_1) T \nu_2(z_2) \inf_{\xi_2 \in \Omega_2(y)} \xi_2(z_2) \\
= \check{\nu}_1(x) T \check{\nu}_2(y) = \check{\nu}_1 \otimes T \check{\nu}_2(x, y).
\] (3.10)

To prove the converse part, we proceed as follows. Since
\[
\check{\nu}_1 \otimes \check{\nu}_2 = \check{\nu}_1 \otimes T \check{\nu}_2 \quad \forall \nu_1 \in I^{G_1}, \nu_2 \in I^{G_2},
\] (3.11)
in view of Theorem 2.10, we have
\[
(\alpha T \nu_1 \otimes T \nu_2)(x) = (\alpha T \check{\nu}_1 \otimes T \check{\nu}_2)(x) \\
= \alpha(x) T (\check{\nu}_1 \otimes T \check{\nu}_2)(x) \\
= \alpha(x) T (\check{\nu}_1(x) T \check{\nu}_2(x)).
\] (3.12)

Since this holds for all \(x\) and for all \(\nu_1\) and \(\nu_2\), with \(\nu_2 = 1\), we have
\[
(\alpha T \nu_1 \otimes T \nu_2)(x) = (\alpha T \check{\nu}_1 \otimes T \nu_2)(x) = \alpha(x) T (\check{\nu}_1(x) T 1) \\
= \alpha(x) T \check{\nu}_1(x) = \alpha(x) T \check{\nu}_1(x) = (\alpha T \nu_1)(x),
\] (3.13)
so \((G_1, -)\) is a \(T\)-neighborhood space. Similarly, with \(\nu_1 = 1\), we see that \((G_2, -)\) is a \(T\)-neighborhood space. This completes the proof. \(\square\)

**Proposition 3.2.** Let \((G_1, -, t(\Omega_1))\) and \((G_2, -, t(\Omega_2))\) be \(T\)-neighborhood spaces. Then the projections

\[
\begin{align*}
pr_1 : (G_1 \times G_2, -\otimes, t(\Omega_1) \otimes_T t(\Omega_2)) &\to (G_1, -, t(\Omega_1)), \quad (x_1, x_2) \mapsto x_1, \\
pr_2 : (G_1 \times G_2, -\otimes, t(\Omega_1) \otimes_T t(\Omega_2)) &\to (G_2, -, t(\Omega_2)), \quad (x_1, x_2) \mapsto x_2,
\end{align*}
\tag{3.14}
\]

are continuous.

**Proof.** Let \(\nu \in \Omega_1(x_1)\) and \(\epsilon > 0\). Then

\[
\begin{align*}
pr_1^{-1}(\nu_1)(x_1, x_2) &= \nu_1(pr_1(x_1, x_2)) = \nu_1(x_1)T1 \geq \nu_1(x_1)Tv_2(x_2) \geq \nu_1 \otimes_T \nu_2(x_1, x_2) - \epsilon \\
&\Rightarrow \nu_1 \otimes_T v_2 - \epsilon \leq pr_1^{-1}(\nu_1) \Rightarrow pr_1^{-1}(\nu_1) \in \Omega(x_1, x_2)^-.
\end{align*}
\tag{3.15}
\]

This implies that \(pr_1 : (G_1 \times G_2, -\otimes, t(\Omega_1) \otimes_T t(\Omega_2)) \to (G_1, -, t(\Omega_1)), \ (x_1, x_2) \mapsto x_1\), is continuous, and similarly, one can prove that \(pr_2 : (G_1 \times G_2, -\otimes, t(\Omega_1) \otimes_T t(\Omega_2)) \to (G_2, -, t(\Omega_2)), \ (x_1, x_2) \mapsto x_2\), is continuous. \(\square\)

**Definition 3.3.** A \(T\)-neighborhood space \((G, -, t(\Omega))\) is said to be a \(TN\)-regular space if and only if for all \(z \in G\), for all \(\mu \in \Omega(z)\), and for all \(\epsilon > 0\), there exists a \(\nu \in \Omega(z)\) closed such that

\[
\epsilon + \mu(z) \geq \inf_{\rho \in \Omega(z)} \sup_{t \in G} \nu(t)T\rho(t) = \check{\nu}(z).
\tag{3.16}
\]

**Theorem 3.4.** Every \(T\)-quasi-uniform space \((G, \Psi)\) is \(TN\)-regular.

**Proof.** Suppose that \(z \in G\), \(\psi \in \Psi\), and \(\epsilon > 0\), and choose \(\psi_\epsilon \in \Psi\) such that

\[
\psi_\epsilon \circ_T \psi_\epsilon \leq \psi + \epsilon.
\tag{3.17}
\]

If \(t \in G\), then by using **Proposition 2.7**,\n
\[
\begin{align*}
\psi_\epsilon(z)^T(t) &= \inf_{\psi_\epsilon' \in \Psi} \sup_{y \in G} \psi_\epsilon(z)(y)T\psi_\epsilon'(y, t) \leq \sup_{y \in G} \psi_\epsilon(z, y)T\psi_\epsilon(y, t) \\
&= \psi_\epsilon \circ_T \psi_\epsilon(z, t) \leq \psi(z, t) + \epsilon \\
&= \psi(z)(t) + \epsilon.
\end{align*}
\tag{3.18}
\]

Hence the result follows. \(\square\)

4. **\(T\)-neighborhood groups.** In what follows, we consider \((G, \cdot)\) as a multiplicative group with \(e\) as the identity element. If \(\mu : G \to I\), then \(\mu^{-1}(x)\) is defined as \(\mu^{-1}(x) = \mu(-x)\), and \(\mu\) is said to be symmetric if and only if \(\mu = \mu^{-1}\)
**Definition 4.1.** Let \((G, \cdot)\) be a group and \((G, -, t(\Omega))\) a \(T\)-neighborhood space with \(T\)-neighborhood base \(\Omega\) on \(G\). Then the quadruple \((G, \cdot, -, t(\Omega))\) is called a \(T\)-neighborhood group if and only if the following properties are satisfied:

\(|TG1|\) the mapping \(m : (G \times G, -^{\otimes_T}, t(\Omega) \otimes_T t(\Omega)) \rightarrow (G, -, t(\Omega)), (x, y) \mapsto xy\), is continuous;

\(|TG2|\) the inversion mapping \(r : (G, -, t(\Omega)) \rightarrow (G, -, t(\Omega)), x \mapsto x^{-1}\), is continuous.

A group structure and a \(T\)-neighborhood system is said to be compatible if and only if \(|TG1|\) and \(|TG2|\) are fulfilled.

**Remarks 4.2.** A \(T\)-neighborhood group may not be a fuzzy topological group in the sense of Foster [8] since we have used \(T\)-neighborhood topology, which differ from the product fuzzy topology.

**Proposition 4.3.** Let \((G, \cdot)\) be a group and \((G, -, t(\Omega))\) a \(T\)-neighborhood space with a \(T\)-neighborhood base \(\Omega\). Then the quadruple \((G, \cdot, -, t(\Omega))\) is a \(T\)-neighborhood group if and only if the mapping

\[
h : (G \times G, -^{\otimes_T}, t(\Omega) \otimes_T t(\Omega)) \rightarrow (G, -, t(\Omega)), (x, y) \mapsto xy^{-1},
\]

is continuous.

**Proof.** Observe that the conditions \(|TG1|\) and \(|TG2|\) are equivalent to the following single condition:

\(|TG3|\) the mapping \(h : (G \times G, -^{\otimes_T}, t(\Omega) \otimes_T t(\Omega)) \rightarrow (G, -, t(\Omega)), (x, y) \mapsto xy^{-1}\), is continuous.

In fact, if we let \(f(x, y) = (x, y^{-1})\), then by \(|TG2|\), \(f\) is continuous and hence in conjunction with \(|TG1|\), one obtains the continuity of \(h\). On the other hand, \(|TG3| \Rightarrow |TG2|\) for \(x \rightarrow ex^{-1} = x^{-1}\) is then continuous; while \(|TG1|\) follows from \(|TG3|\) and \(|TG2|\), because \((x, y) \rightarrow x(y^{-1})^{-1} = xy\) is then continuous. □

**Proposition 4.4.** Let \((G, \cdot)\) be a group and \((G, -, t(\Omega))\) a \(T\)-neighborhood space with \(\Omega\) a \(T\)-neighborhood base in \(G\). Then

(a) the mapping \(m : (G \times G, -^{\otimes_T}, t(\Omega) \otimes_T t(\Omega)) \rightarrow (G, -, t(\Omega)), (x, y) \mapsto xy\), is continuous at \((e, e) \in G \times G\) if and only if for all \(\mu \in \Omega(e)\), for all \(\varepsilon > 0\), there exists \(\nu \in \Omega(e)\) such that

\[
\nu \otimes_T \varepsilon \leq \mu + \varepsilon;
\]

(b) the inversion mapping \(r : (G, -, t(\Omega)) \rightarrow (G, -, t(\Omega)), x \mapsto x^{-1}\), is continuous at \(e \in G\) if and only if for all \(\mu \in \Omega(e)\), for all \(\varepsilon > 0\), there exists \(\nu \in \Omega(e)\) such that

\[
\nu \leq \mu^{-1} + \varepsilon.
\]

**Proof.** (a) In view of Theorem 2.4, continuity at \((e, e) \in G \times G\) is equivalent to

\[
\forall \mu \in \Omega(e) \implies m^{-1}(\mu) \in (\Omega(e) \otimes_T \Omega(e))^-
\]

\[
\iff \forall \mu \in \Omega(e), \forall \varepsilon > 0, \exists \nu = \nu_\varepsilon \in \Omega(e) \ni \nu \otimes_T \varepsilon \leq m^{-1}(\mu) + \varepsilon.
\]
But \( m(\nu \otimes_T \nu)(z) = \sup_{(x,y) \in m^{-1}(z)} \nu(x)T\nu(y) = \sup_{xy = z} \nu(x)T\nu(y) = \nu \otimes_T \nu(z). \) Thus, in this case, continuity at \((e,e) \in G \times G\) is in fact equivalent to

\[
\nu \otimes_T \nu \leq \mu + \epsilon. \tag{4.5}
\]

(b) This follows almost in the same way as in (a).

**Corollary 4.5.** If \((G, \cdot, t(\Omega))\) is a \(T\)-neighborhood group, then the mapping \((4.1)\) is continuous at \((e,e) \in G \times G\) if and only if for all \(\mu \in \Omega(e)\) and for all \(\epsilon > 0\), there exists a \(\nu \in \Omega(e)\) such that

\[
\nu \otimes_T \nu^{-1} \leq \mu + \epsilon. \tag{4.6}
\]

**Proof.** This follows at once from the composition of (a) and (b) in Proposition 4.4.

**Proposition 4.6.** Let \((G, -, t(\Omega))\) be a \(T\)-neighborhood space and \(A \subset G\). Then \((A, -, t(\Omega|_A))\) is a \(T\)-neighborhood space, a subspace of the \(T\)-neighborhood space \((G, -, t(\Omega))\).

**Proof.** The proof follows by easy verification.

**Theorem 4.7.** The triple \((G, \cdot, \tau)\) is a topological group if and only if the quadruple \((G, \cdot, -, t(\Omega))\), where \(\Omega\) is the generated \(T\)-neighborhood basis, is a \(T\)-neighborhood group.

**Proof.** With the help of Proposition 2.9, it follows that the mapping

\[
h : (G \times G, \mathcal{V}_\tau \times \mathcal{V}_\tau) \to (G, \mathcal{V}_\tau), \quad (x, y) \mapsto xy^{-1}, \tag{4.7}
\]

is continuous if and only if

\[
h : (G \times G, t(\Omega) \otimes_T t(\Omega)) \to (G, t(\Omega)), \quad (x, y) \mapsto xy^{-1}, \tag{4.8}
\]

is continuous, where \(\Omega\) is the basis for the generated \(T\)-neighborhood spaces.

**Lemma 4.8.** Let \((G, \cdot, -, t(\Omega))\) be a \(T\)-neighborhood group and \(a \in G\). Then

1. the left translation \(L_a : (G, \cdot, -, t(\Omega)) \to (G, \cdot, -, t(\Omega)), x \mapsto ax,\) and the right translation \(R_a : (G, \cdot, -, t(\Omega)) \to (G, \cdot, -, t(\Omega)), x \mapsto xa,\) are homeomorphisms;
2. the inner automorphism \(I_a : (G, \cdot, -, t(\Omega)) \to (G, \cdot, -, t(\Omega)), z \mapsto az\) is an isomorphism;
3. \(\nu \in \Omega(e)\) if and only if \(L_a(\nu) \in \Omega(a)\) if and only if \(R_a(\nu) \in \Omega(a)\). In other words, if \(\Omega\) is saturated, then \(\nu \in \Omega(e)\) if and only if \(1_{[a]} \otimes_T \nu = a \otimes_T \nu \in \Omega(a)\) if and only if \(\nu \otimes_T a \in \Omega(a)\);
4. \(\nu \in \Omega(a)\) if and only if \(L_{-a}(\nu) \in \Omega(e)\) if and only if \(R_{-a}(\nu) \in \Omega(e)\). In other words, if \(\Omega\) is saturated, then \(\nu \in \Omega(a)\) if and only if \(1_{[-a]} \otimes_T \nu = a^{-1} \otimes_T \nu \in \Omega(e)\) if and only if \(\nu \otimes_T a^{-1} \in \Omega(e)\);
5. if \(\nu \in \Omega(e)\), then \(\nu^{-1} \in \Omega(e)\);
6. \(\nu \otimes_T \nu^{-1}\) is symmetric.
Conversely, let \( \mathcal{L}_a \circ \mathcal{R}_{-a} = \mathcal{R}_{-a} \circ \mathcal{L}_a \) for all \( a \in G \).

Let \( \nu \in \Omega(e)^- \subseteq \Omega(e) \), that is, \( \nu \in \Omega(a^{-1}a) = \Omega(\mathcal{L}_a^{-1}(a)) \). Since \( \mathcal{L}_a^{-1} \) is continuous, then in view of Theorem 2.4, \( \mathcal{L}_a(v) = (\mathcal{L}_a^{-1})(v) \in \Omega(a)^- \) implies \( \mathcal{L}_a(v) \in \Omega(a)^- \). Conversely, let \( \mathcal{L}_a(v) \in \Omega(a)^- \subseteq \Omega(a) \) implies \( \mathcal{L}_a(v) \in \Omega(a) = \Omega(\mathcal{L}_a(e)) \), and since \( \mathcal{L}_a : G \to G \) is continuous injection again by Theorem 2.4, \( v = \mathcal{L}_a^{-1}(\mathcal{L}_a(v)) \in \Omega(e)^- \). For the calculations of the other part, see [7, Theorem 5.1.1].

(4) follows from (3) while (5) follows from the fact that the inversion mapping \( r : G \to G, x \mapsto x^{-1} \) is a homeomorphism.

(6) We have \( \nu \otimes_T \nu^{-1} = (\nu \otimes_T \nu^{-1})^{-1} \). If \( x \in G \), then

\[
(v \otimes_T \nu^{-1})^{-1}(x) = (v \otimes_T \nu^{-1})(x^{-1}) = \sup_{ab=x^{-1}} v(a)T\nu(b^{-1})
\]

\[
= \sup_{st^{-1}=x^{-1}} v(s)T\nu(t) = \sup_{ts^{-1}=x} v(t)T\nu(s)
\]

\[
= \sup_{ts^{-1}=x} v(t)T\nu((s^{-1})^{-1}) = \sup_{ts^{-1}=x} v(t)T\nu^{-1}(s^{-1})
\]

\[
= v \otimes_T \nu^{-1}(x).
\]

This completes the proof. \( \square \)

**Definition 4.9.** A \( T \)-neighborhood space \( (G, \cdot, t(\Omega)) \) is called homogeneous space if and only if for all \( (a, b) \in G \times G \), there exists a homeomorphism \( f : (G, \cdot, t(\Omega)) \to (G, \cdot, t(\Omega)) \) such that \( f(a) = b \).

**Theorem 4.10.** Every \( T \)-neighborhood group is a homogeneous space.

**Proof.** This follows from the fact that for all \( a, b \in G \times G \), the function

\[
\mathcal{R}_{a^{-1}b} : G \to G,
\]

\[
x \mapsto xa^{-1}b,
\]

is a homeomorphism. \( \square \)

**Lemma 4.11.** Let \( (G, \cdot) \) be a group, and let, for all \( \mu \in I^G \), \( \mu_L : G \times G \to I \), \( (x, y) \mapsto \mu_L(x, y) = \mu(x^{-1}y) \) (resp., \( \mu_R : G \times G \to I \), \( (x, y) \mapsto \mu_R(x, y) = \mu(yx^{-1}) \)) be the vicinities \( L \)-associated (resp., \( R \)-associated) with \( \mu \).

Then for all \( \mu, \theta, \nu \in I^G \), \( (x, y) \in G \times G \), and triangular norm \( T : I \times I \to I \), the following hold:

1. \( \mu_L(\theta)_T = \theta \otimes_T \mu \) (resp., \( \mu_R(\theta)_T = \mu \otimes_T \theta \));
2. \( \mu_L T\nu_L = (\mu T\nu)_L \) (resp., \( \mu_R T\nu_R = (\mu T\nu)_R \));
3. \( (\mu_L^T)^T = (\mu_L^T)^T \);
4. \( \mu_L \otimes_T \nu_L = (\nu \otimes_T \mu)_L \) (resp., \( \mu_R \otimes_T \nu_R = (\nu \otimes_T \mu)_R \)).

**Proof.** (1) For all \( (x, \theta, \mu) \in G \times I^G \times I^G \),

\[
\mu_L(\theta)_T(x) = \sup_{y \in G} [\theta(y)T\mu_L(y, x)] = \sup_{y \in G} [\theta(y)T\mu(y^{-1}x)]
\]

\[
= \theta \otimes_T \mu(x) \quad (\text{by} \ [7, \text{Theorem 5.1.1}]).
\]
(2) and (3) are obvious.

(4) For all \((x, y) \in G \times G\),

\[
\mu_L \odot_T \nu_L(x, y) = \sup_{z \in G} [\nu_L(x, z) T \mu_L(z, y)] = \sup_{z \in G} [\nu(x^{-1} z) T \mu(z^{-1} y)]
\]

\[
= \sup_{st = x^{-1} z z^{-1} y = x^{-1} y} [\nu(s) T \mu(t)] = (\nu \odot_T \mu)(x^{-1} y)
\]

\[
= (\nu \odot_T \mu)_L(x, y).
\]

\[
(4.12)
\]

**Theorem 4.12.** Every \(T\)-neighborhood group is a \(T\)-uniform space.

**Proof.** If \((G, \cdot, t(\Omega))\) is a \(T\)-neighborhood group, then \((G, \cdot, t(\Omega))\) is a \(T\)-neighborhood space with the \(T\)-neighborhood basis \(\Omega\).

We consider the following collection:

\[
\Omega = \{\mu_L \mid \mu \in \Omega(e)\} \subset IG \times G.
\]

(4.13)

We claim that \(\Omega\) is a \(T\)-uniform basis.

(TUB1) Clearly \(\Omega\) is a prefilter basis.

(TUB2) If \(\psi \in \Omega\), then there exists a \(\mu \in \Omega(e)\) such that \(\psi = \mu_L\), and for all \(x \in G\),

\[
\psi(x, x) = \mu_L(x, x) = \mu(e) = 1.
\]

(4.14)

(TUB3) If \(\psi \in \Omega\), then there exists a \(\mu \in \Omega(e)\) such that \(\psi = \mu_L\).

Thus, by virtue of Proposition 4.4(a), for all \(\epsilon > 0\), we can find \(\nu^\epsilon \in \Omega(e)\) such that

\[
\nu^\epsilon \odot_T \nu^\epsilon - \epsilon \leq \mu.
\]

(4.15)

If we let \(\nu^\epsilon_L = \psi^\epsilon\), then one obtains

\[
\psi^\epsilon \odot_T \psi^\epsilon - \epsilon = \nu^\epsilon_L \odot_T \nu^\epsilon_L - \epsilon = (\nu^\epsilon \odot_T \nu^\epsilon)_L - \epsilon \leq \mu_L
\]

\[
\Rightarrow \psi^\epsilon \odot_T \psi^\epsilon - \epsilon \leq \psi.
\]

(4.16)

(TUB4) If \(\psi \in \Omega\), then there is a \(\mu \in \Omega(e)\) such that \(\psi = \mu_L\). Consequently, by Proposition 4.4(b), for all \(\epsilon > 0\), there exists a \(\nu^\epsilon \in \Omega(e)\) such that

\[
\nu^\epsilon - \epsilon \leq \mu^{-1}.
\]

(4.17)

Therefore, \(\nu^\epsilon_L - \epsilon \leq (\mu^{-1})_L = (\mu_L)^{-1}\) implies \(\psi^\epsilon - \epsilon \leq \psi_s\).

This shows in accordance with Definition 2.5 that \(\Omega\) is a \(T\)-uniform basis, which in turn gives rise to a left \(T\)-uniformity \(\mathcal{U}_L = \Omega^-\).

In fact, we have for all \(x \in G\),

\[
\mathcal{U}_L(x) = \{\mu_L(1_x) \mid \mu \in \Omega(e)\}^- = \{\mathcal{L}_x(\mu) \mid \mu \in \Omega(e)^-\} = \Omega(x)^-,
\]

(4.18)

which is a \(T\)-neighborhood system on \(G\) and that \((G, t(\Omega) = t(\mathcal{U}_L))\) is a \(T\)-uniform space. Similarly, one can obtain right \(T\)-uniformity.

**Theorem 4.13.** Every \(T\)-neighborhood group is \(T\)-completely regular.
PROOF. This follows from the preceding theorem in conjunction with Theorem 2.12 because every $T$-neighborhood group is $T$-uniformizable and every $T$-uniformizable space is $T$-completely regular.

**Theorem 4.14.** Let $(G, \cdot)$ be a group, $(G, -, t(\Omega))$ a $T$-neighborhood space with $T$-neighborhood base $\Omega$ in $G$. Then the quadruple $(G, \cdot, -, t(\Omega))$ is a $T$-neighborhood group if and only if the following are true:

1. for all $a \in G, \Omega(a)^- = \{L_a(\mu) \mid \mu \in \Omega(e)^-\}$ (resp., for all $a \in G, \Omega(a)^- = \{R_a(\mu) \mid \mu \in \Omega(e)^-\});
2. for all $\mu \in \Omega(e)$, for all $\epsilon > 0$, there exists $v \in \Omega(e)$ such that
   
   \[ v \circ_T v \leq \mu + \epsilon, \]
   
   that is, the mapping $m : (x, y) \mapsto xy$ is continuous at $(e, e) \in G \times G$;
3. for all $\mu \in \Omega(e)$, for all $\epsilon > 0$, there exists $v \in \Omega(e)$ such that
   
   \[ v \leq \mu^{-1} + \epsilon, \]
   
   that is, the mapping $\nu : x \mapsto x^{-1}$ is continuous at $e \in G$;
4. for all $\mu \in \Omega(e)$, for all $\epsilon > 0$, and for all $a \in G$ such that
   
   \[ a \circ_T \nu \circ_T a^{-1} \leq \mu + \epsilon, \]
   
   that is, the mapping $\delta_a : x \mapsto axa^{-1}$ is continuous at $e \in G$.

**Proof.** Let $(G, \cdot, -, t(\Omega))$ be a $T$-neighborhood group. Then the conditions (1), (2), (3), and (4) are clearly true.

To prove the converse part, we remark that from Corollary 4.5, it follows that the mapping $h : G \times G \to G; (x, y) \mapsto xy^{-1}$ is continuous at $(e, e)$, and since the translations $L_a$ and $R_a$ are continuous at $a$ and $e$, respectively, the continuity of $m$ follows from the following chain:

\[
G \times G \xrightarrow{y_a^{-1} \times y_b^{-1}} G \times G \xrightarrow{m} G \xrightarrow{\delta_b} G \xrightarrow{y_{ab}^{-1}} G,
\]

where $(a, b) \to (e, e) \to e \to e \to ab^{-1}$.

**Theorem 4.15.** Let $(G, \cdot)$ be a group and $\mathcal{F}$ a collection of nonempty subsets of $I^G$, that is, $\emptyset \neq \mathcal{F} \subset I^G$ such that

1. $\mathcal{F}$ is a prefilterbasis and $\mu(e) = 1$ for all $\mu \in \mathcal{F}$;
2. for all $\mu \in \mathcal{F}$, for all $\epsilon > 0$, there exists $v \in \mathcal{F}$ such that $v - \epsilon \leq \mu^{-1}$;
3. for all $\mu \in \mathcal{F}$, for all $\epsilon > 0$, there exists $v \in \mathcal{F}$ such that $v \circ_T v - \epsilon \leq \mu$;
4. for all $\mu \in \mathcal{F}$, for all $a \in G$, for all $\epsilon > 0$, there exists $v \in \mathcal{F}$ such that $a \circ_T v \circ_T a^{-1} - \epsilon \leq \mu$.

Then there exists a unique $T$-neighborhood system compatible with the group structure of $G$ such that $\mathcal{F}$ is a $T$-neighborhood basis at $e \in G$.

**Proof.** For all $\mu \in \mathcal{F}$, let $\mu_L : G \times G \to I$ be the vicinities $L$-associated with $\mu$. Evidently, $\mu_L(a, a) = \mu(a^{-1}a) = \mu(e) = 1$. 
We let
\[ \mathcal{B} = \{ \mu_L \mid \mu \in \mathcal{F} \} \subset I^{G \times G}. \] (4.23)

We show that \( \mathcal{B} \) is a \( T \)-quasi-uniform basis for a \( T \)-quasi-uniformity. We verify Definition 2.5 upto (TUB3).

(TUB1) \( \mathcal{B} \) is a prefilter basis; for \( 0 \notin \mathcal{B} \) which is clearly true, since \( \mathcal{F} \) is a prefilter basis.

Next, let \( \lambda, \xi \in \mathcal{B} \), then \( \lambda = \mu_L \) for some \( \mu \in \mathcal{F} \) and \( \xi = \eta_L \) for some \( \eta \in \mathcal{F} \). Since \( \mathcal{F} \) is a prefilter basis, there exists a \( \theta \in \mathcal{F} \) such that \( \theta \leq \mu \land \eta \) and \( \theta_L = \mu_L \land \eta_L = \lambda \land \xi \), proving that \( \mathcal{B} \) is indeed a prefilter basis.

(TUB2) For all \( x \in G \), and \( \psi \in \mathcal{B} \), we have \( \psi = \mu_L \) for some \( \mu \in \mathcal{F} \) and \( \psi(x,x) = \mu_L(x,x) = \mu(e) = 1 \) by (1).

(TUB3) Let \( \psi \in \mathcal{B} \). Then there exists a \( \mu \in \mathcal{F} \) such that \( \psi = \mu_L \).

Now by (3), for all \( \epsilon > 0 \), we can find a \( \nu \in \mathcal{F} \) such that
\[ \nu \odot \nu^* - \epsilon \leq \mu. \] (4.24)

But then by virtue of Lemma 4.11(4), we get \( \nu_L \odot \nu_L^* - \epsilon \leq \mu_L \). So, if we put \( \psi_\epsilon = \nu_L \), then
\[ \psi_\epsilon \odot \psi_\epsilon^* - \epsilon \leq \psi. \] (4.25)

This completes the proof that \( \mathcal{B} \) is a \( T \)-quasi-uniform basis which in turn gives rise to a \( T \)-quasi-uniformity and hence a \( T \)-quasi-uniform space. Then in view of the Theorem 2.6, since every \( T \)-quasi-uniform space is a \( T \)-neighborhood space, in this case, we have the \( T \)-neighborhood system as given by the family
\[ \{ \mu_L(1x)_{\mathcal{T}} \mid \mu_L \in \mathcal{B}^\sim \} = \{ \mu_L(1x)_{\mathcal{T}} \mid \mu_L \in \mathcal{B} \}^\sim \]
\[ = \{ 1x \odot \mathcal{T} \mu \mid \mu \in \mathcal{F} \}^\sim \]
\[ = \{ 1x \odot \mu \mid \mu \in \mathcal{F} \}^\sim. \] (4.26)

Thus one obtains the \( T \)-neighborhood system with the following family: \( \Omega(x) = \{ 1x \odot \mathcal{T} \mu \mid \mu \in \mathcal{F} \} \), a basis for the system in question.

**Theorem 4.16.** Let \( (G, \cdot, ^{-}, t(\Omega)) \) be a \( T \)-neighborhood group. Then for all \( \mu : G \to I \),
\[ \bar{\mu} = \inf \{ \mu \odot \nu \mid \nu \in \Omega(e)^\sim \} = \inf \{ \mu \odot \nu \mid \nu \in \Omega(e) \}^\sim. \] (4.27)

**Proof.** Observe that every \( T \)-neighborhood group is a \( T \)-quasi-uniform space. Therefore, by virtue of Theorem 2.6, we can write, in particular, that
\[ \bar{\mu} = \inf \{ \nu_L(\mu)_{\mathcal{T}} \mid \nu \in \Omega(e)^\sim \}. \] (4.28)

Then by using Lemma 4.11(1), we have the following:
\[ \bar{\mu} = \inf \{ \mu \odot \nu \mid \nu \in \Omega(e)^\sim \} = \inf \{ \mu \odot \nu \mid \nu \in \Omega(e) \}^\sim. \] (4.29)
Corollary 4.17. In a $T$-neighborhood group $(G, \cdot, t(\Omega))$, the following property holds:

$$\bar{\mu} = \inf \{ v \circ_T \mu \mid v \in \Omega(e)^{-} \} = \inf \{ v \circ_T \mu \mid \mu \in \Omega(e) \}^{-}. \quad (4.30)$$

Proof. This follows at once from the preceding results.

Theorem 4.18. If $(G, \cdot, t(\Omega))$ is a $T$-neighborhood group, then $(G, t(\Omega))$ is $T$-regular.

Proof. Let $\mu \in \Omega(e)$ and $\epsilon > 0$. Since the map $(x, y) \mapsto x y^{-1}$ is continuous at $(e, e) \in G \times G$, in view of Corollary 4.5, we can find a $v \in \Omega(e)$ such that

$$v \circ_T v^{-1} \leq \mu + \epsilon. \quad (4.31)$$

Then using Theorem 4.16, we obtain

$$\bar{\mu}(x) = \inf_{\omega \in \Omega(e)} v \circ_T \omega^{-1} \leq v \circ_T v^{-1} \leq \mu(x) + \epsilon, \quad (4.32)$$

which ends the proof.

Proposition 4.19. If $(G, \cdot, t(\Omega))$ is a $T$-neighborhood group, then for all $\mu, \nu \in I_{G}$, we have the following:

(i) $\bar{\mu} \circ_T \bar{\nu} \leq \bar{\mu} \circ_T \bar{\nu}$;
(ii) $\mu^{-1} = \bar{\mu}^{-1}$;
(iii) $x \circ_T \mu \circ_T y = x \circ_T \bar{\mu} \circ_T y$ for all $x, y \in G$.

Proof. (i) If $z \in G$, then we have

$$\bar{\mu} \circ_T \bar{\nu}(z) = \sup_{\omega = z} \bar{\mu}(x) \circ_T \bar{\nu}y = \sup_{(x, y) \in m^{-1}(z)} [\bar{\mu} \circ_T \bar{\nu}](x, y)$$

$$= m[\bar{\mu} \circ_T \bar{\nu}](z) = m[\bar{\mu} \circ_T \bar{\nu}](z) \quad (4.33)$$

(ii) and (iii) follow immediately.

Lemma 4.20. If $(G, \cdot)$ and $(G', \cdot)$ are groups and $f : G \to G'$ is a group homomorphism, then

$$f(x \circ_T a^{-1} \circ_T \mu) = f(x) \circ_T f(a)^{-1} \circ_T f(\mu). \quad (4.34)$$

Proof. This follows the same way as in [2, Lemma 2.15]; see also [7].

Theorem 4.21. Let $(G, \cdot, t(\Omega))$ and $(H, \cdot, t(\Xi))$ be $T$-neighborhood groups with bases $\Omega$ and $\Xi$ in $G$ and $H$, respectively. If $f : G \to H$ is a group homomorphism, then $f$ is continuous if and only if it is continuous at one point.

Proof. Let $f : G \to H$ be continuous at the point $a \in G$. We need to show that $f$ is continuous at each $x \in G$. Let $\xi \in \Xi(f(x))$ and $\epsilon > 0$. Then we have $f(x)^{-1} \circ_T \xi \in \Xi(e)$ and hence $f(a) \circ_T f(x)^{-1} \circ_T \xi \in \Xi(f(a))$. Then by Theorem 2.4, the continuity at one...
point \( a \in G \) yields that \( f^{-1}(f(a) \otimes_T f(x)^{-1} \otimes_T \xi) \in \Omega(a)^{-} \), which in turn implies that there exists a \( \sigma = \sigma_{\varepsilon} \in \Omega(a) \) such that

\[
\sigma - \varepsilon \leq f^{-1}(f(a) \otimes_T f(x)^{-1} \otimes_T \xi). \tag{4.35}
\]

Now we have \( \mu := x \otimes_T a^{-1} \otimes_T \sigma \in \Omega(x) \).

Thus, one obtains

\[
\mu(z) - \varepsilon = x \otimes_T a^{-1} \otimes_T \sigma(z) - \varepsilon = \sigma(ax^{-1}z) - \varepsilon
\leq f(a) \otimes_T f(x)^{-1} \otimes_T \xi(f(ax^{-1}z))
= f(ax^{-1}) \otimes_T \xi(f(ax^{-1}f(z))
= \xi(f(z)) = f^{-1}(\xi)(z), \tag{4.36}
\]

that is, \( \mu - \varepsilon \leq f^{-1}(\xi) \), which implies that \( f^{-1}(\xi) \in \Omega(x)^{-} \).

Now we present some results on \( T \)-neighborhood groups in conjunction with Mordeson’s \( TI \)-group.

5. Application of \( T \)-neighborhood groups in \( TI \)-groups

**Definition 5.1 [7, 20]**. An \( I \)-subset \( \mu \) of \( G \) is called a \( TI \)-subgroup of \( G \) if it fulfills the following conditions:

\( G_1 \) \( \mu(e) = 1; \)
\( G_2 \) \( \mu(x^{-1}) \geq \mu(x) \), for all \( x \in G; \)
\( G_3 \) \( \mu(xy) \geq \mu(x)T\mu(y) \), for all \( x, y \in G. \)

We denote the set of all \( TI \)-subgroups of \( G \) by \( TI(G) \) and that of the set of all normal \( TI \)-subgroups by \( NTI \)-subgroups, while by \( NI \)-subgroup we mean normal \( I \)-subgroups, the one introduced by Rosenfeld [20] in which case \( T = \min \) is used.

**Proposition 5.2**. Let \((G, \cdot, -, t(\Omega))\) be a \( T \)-neighborhood group and \( \mu \in TI(G) \). Then \( \tilde{\mu}^{t(\Omega)} \in TI(G) \).

**Proof**. In view of [7, Theorem 5.1.4], it suffices to prove that

\[
\tilde{\mu}^{t(\Omega)} \otimes_T (\tilde{\nu}^{t(\Omega)})^{-1} \leq \tilde{\mu}^{t(\Omega)} \tag{5.1}
\]

Since \( \mu \in TI(G) \), we have \( \mu \otimes_T \mu^{-1} \leq \mu \). Then by an easy calculation, one obtains

\[
\mu \otimes_T \mu^{-1} = h(\mu \otimes_T \mu^{-1}), \tag{5.2}
\]

which in conjunction with **Theorem 3.1**, yields the following:

\[
(\tilde{\mu}^{t(\Omega)}) \otimes_T (\tilde{\mu}^{t(\Omega)})^{-1} = h[(\tilde{\mu}^{t(\Omega)}) \otimes_T (\tilde{\mu}^{t(\Omega)})^{-1}] \leq \tilde{\mu}^{t(\Omega)}, \tag{5.3}
\]

which proves that \( \tilde{\mu}^{t(\Omega)} \in TI(G) \).
Proposition 5.3. If \((G, \cdot, ^{-}, t(\Omega))\) is a \(T\)-neighborhood group and \(\mu \in NTI(G)\), then \(\bar{\mu}t(\Omega) \in NTI(G)\).

Proof. Since \(\mu \in NI(G)\), we have \(I_x(\mu) = x \circ_T \mu \circ_T x^{-1} = \mu\), where \(I_x : G \to G\), \(z \mapsto xzx^{-1}\) is an inner automorphism. But then using Proposition 4.19(iii), we obtain
\[
x \circ_T \bar{\mu} \circ_T x^{-1} = \overline{x \circ_T \mu \circ x^{-1}} = \bar{\mu}.
\]
(5.4)
Hence the result follows from [7, Theorem 5.2.1(N5)].

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