SOME ANALYTICAL PROPERTIES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF NONINTEGER ORDER

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The analytical properties of solutions of the nonlinear differential equations $x^{(\alpha)}(t) = f(t, x)$, $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$ of noninteger order have been investigated. We obtained two results concerning the frame curves of solutions. Moreover, we proved a result on differential inequality with fractional derivatives.

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1. Introduction. The problem of existence and uniqueness of solutions of the nonhomogeneous differential equations with fractional derivatives

$$x^{(\alpha)}(t) = f(t, x), \quad \alpha \in \mathbb{R}, \quad 0 < \alpha \leq 1,$$

with the initial condition

$$x^{(\alpha-1)}(t_0) = x_0,$$

where $\mathbb{R}$ is the set of real numbers, $t \in I = [0, \infty)$, and $f$ is a real-valued function on $D = I \times \mathbb{R}^n$ into $\mathbb{R}^n$ where $\mathbb{R}^n$ denotes the real $n$-dimensional Euclidean space, and $x_0 \in \mathbb{R}^n$, has been investigated by some authors (see [1, 2, 6, 9]).

In recent years, interest has increased concerning the numerical treatment of fractional differential equations (see [4, 5, 11, 12]). On the other hand, differential inequalities and comparison theorems with the unique solution are very important for the numerical solution of differential equations (see [8] for fractional differential equations, and [10] for ordinary differential equations).

In this note, we will obtain a differential inequality result of (1.1) and (1.2), our result is more general than that in [8]. Also, we obtain two results concerning frame curves, the lower and upper frame curves of the solutions of (1.1) and (1.2); these two results are extensions to those in [10] for ordinary differential equations.

We will use the definitions and terminology used in Barrett [3] and Al-Bassam [2].

It is worth mentioning that it was shown by Hadid and Alshamani [7] that the solutions of (1.1) and (1.2) satisfy the integral equation

$$x(t) = \frac{x_0(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s, x(s)) ds,$$

(1.3)
where \(0 < t_0 < t \leq t_0 + a\), provided that the integral exists in the Lebesque sense, where \(\Gamma\) is the Gamma function.

2. The main theorems. In this section, we will prove the main theorems.

**Theorem 2.1.** Let \(f(t,x)\) be a continuous function on the region
\[
\mathbb{R}(a,b) : 0 < t_0 < t \leq t_0 + a, \quad |x - x_0(t - t_0)^{\alpha - 1}| \leq b.
\] (2.1)

Suppose \(x_1(t)\) is a solution of the differential inequality
\[
x^{(\alpha)}(t) \leq f(t,x_1(t)) \quad \text{on} \quad (t_0,t_0 + a],
\] (2.2)

then there exists a solution \(x_2(t)\) of the differential inequality
\[
x^{(\alpha)}_2(t) \geq f(t,x_2(t)) \quad \text{on} \quad (t_0,t_0 + a], \quad x^{(\alpha - 1)}_1(t_0) \leq x^{(\alpha - 1)}_2(t_0)
\] (2.3)
such that on this interval, \(x_1(t) \leq x_2(t)\).

**Proof.** Let \(\psi(t,x_2) = f(t,\max(x_2,x_1(t)))\). Obviously, \(\psi\) is a continuous function on \(\mathbb{R}\).

First we will prove the inequality
\[
x_1(t) \leq w(t) \quad \text{for} \quad t \in [t_0,t_0 + a],
\] (2.4)

where \(w(t)\) satisfies the differential inequality
\[
w^{(\alpha)}(t) \geq \psi(t,w(t)), \quad w^{(\alpha - 1)}(t_0) = x^{(\alpha - 1)}_2(t_0).
\] (2.5)

Suppose that this is not true, that is, that for some value \(\tau\), \(w(\tau) < x_1(\tau)\). Let \(\tau_0\) be the lower bound of numbers \(s\) for which we have \(w(t) < x_1(t)\) for \(s \leq t \leq \tau\). Then \(w(\tau_0) = x_1(\tau_0)\) and \(w(t) < x_1(t)\) for \(\tau_0 < t < \tau\).

Therefore, we get
\[
\psi(t,w(t)) = f(t,x_1(t)) \quad \text{on} \quad t \in [\tau_0,\tau].
\] (2.6)

Using inequality (2.2) and (1.3), it follows that
\[
x_1(\tau) \leq \frac{x_1(\tau_0)(\tau - \tau_0)^{\alpha - 1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_{\tau_0}^{\tau} (\tau - s)^{\alpha - 1} f(s,x_1(s)) ds \quad (2.7)
\]

and from (2.5) and (2.7), we get \(x_1(\tau) \leq w(\tau)\), which is in contradiction with our supposition. This proves inequality (2.4).

But now because (2.4) implies that a solution of inequality (2.5) is also a solution of inequality (2.3), we see that the result follows from (2.4).

**Remark 2.2.** The above theorem means that the solution \(x_1(t)\) is dominated by the solution \(x_2(t)\). Moreover, if \(x_2(t)\) is a bounded solution, then so is \(x_1(t)\).
**Theorem 2.3.** Let $\phi(t, y), f(t, y),$ and $F(t, y)$ be continuous functions on the region 
\[ R_1(a, b) : 0 < t_0 < t \leq t_0 + a, \quad |y - y_0(t - t_0)^{\alpha - 1}| \leq b \]  
(2.8) 
and satisfy
\[ \phi(t, y) \leq f(t, y) \leq F(t, y). \]  
(2.9)

Further let $x = x(t), y = y(t),$ and $X = X(t)$ be solutions of the differential equations
\[ x^{(\alpha)}(t) = \phi(t, x), \quad y^{(\alpha)}(t) = f(t, y), \quad X^{(\alpha)}(t) = F(t, X), \]  
(2.10)

which pass through the point $(t_0, y_0(t - t_0)^{\alpha - 1})$, defined on $[t_0, t_0 + a]$, and which lie between $y_0(t - t_0)^{\alpha - 1} - b$ and $y_0(t - t_0)^{\alpha - 1} + b$.

If the function $f(t, y)$ satisfies the Lipschitz condition in the second parameter on $R_1(a, b)$:
\[ |f(t, y_1) - f(t, y_2)| \leq L |y_1 - y_2| \]  
(2.11)
for some positive constant $L$, then
\[ x(t) \leq y(t) \leq X(t). \]  
(2.12)

**Proof.** It is clear from Theorem 2.1 and equations (2.9) and (2.10) that the following inequalities:
\[ X^{(\alpha)}(t) - f(t, X) \geq 0, \quad x^{(\alpha)}(t) - f(t, x) \leq 0, \quad x(t) \leq y(t) \leq X(t), \]  
(2.13)
are satisfied if
\[ X = y_0(t - t_0)^{\alpha - 1} + Y, \quad x = y_0(t - t_0)^{\alpha - 1} - Y, \]  
(2.14)
where
\[ Y = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} |f(s, y_0)| \, ds \leq \frac{M}{\alpha \Gamma(\alpha)} (t - t_0)^{\alpha}, \]  
(2.15)
and $M$ is a positive real constant such that $|f(s, y)| \leq M$.

Hence, the theorem is proved. \(\square\)

**Remark 2.4.** The functions $X(t)$ and $x(t)$ are called “frame curves.”

**Theorem 2.5.** Let the functions $f(t, y), F(t, y), y(t),$ and $X(t)$ be defined as in Theorem 2.3. Set $h(t) = X^{(\alpha)}(t) - f(t, X(t))$, then the function
\[ X_1(t) = X(t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} e^{-L(t-s)} h(s) \, ds, \]  
(2.16)

where $L$ the Lipschitz constant for the function $f(t, y)$ is an upper frame curve on the interval $[t_0, t_0 + a]$, and on that interval there exist the inequalities
\[ y(t) \leq X_1(t) \leq X(t). \]  
(2.17)
Proof. The inequality $X_1(t) \leq X(t)$ is obvious. On the other hand, as in [10], we have

$$X^{(\alpha)}(t) - f(t, X_1(t)) = X^{(\alpha)}(t) - h(t) + \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-L(t-s)} h(s) ds - f(t, X_1(t))$$

(2.18)

$$= f(t, X_1(t)) - f(t, X(t)) + L[X(t) - X_1(t)] \geq 0.$$

Thus $y(t) \leq X_1(t)$.

Remark 2.6. By using the same above procedure, we can show that if $x(t)$ a lower frame and if we set

$$h_1(t) = x^{(\alpha)}(t) - f(t, x(t)),$$

(2.19)

then

$$x_1(t) = x(t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-L(t-s)} h_1(s) ds$$

(2.20)

is also a lower frame curve and we have

$$y(t) \leq x_1(t) \leq y(t).$$

(2.21)

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References


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