SKEW-SYMMETRIC VECTOR FIELDS ON A CR-SUBMANIFOLD OF A PARA-KÄHLERIAN MANIFOLD

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We deal with a CR-submanifold $M$ of a para-Kählerian manifold $\tilde{M}$, which carries a $J$-skew-symmetric vector field $X$. It is shown that $X$ defines a global Hamiltonian of the symplectic form $\Omega$ on $M^\top$ and $JX$ is a relative infinitesimal automorphism of $\Omega$. Other geometric properties are given.

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1. Introduction. CR-submanifolds $M$ of some pseudo-Riemannian manifolds $\tilde{M}$ have been first investigated by Rosca [10], and also studied in [2, 3, 11].

If $\tilde{M}$ is a para-Kählerian manifold, it has been proved that any coisotropic submanifold $M$ of $\tilde{M}$ is a CR-submanifold (such CR-submanifolds have been denominated CICR-submanifolds [6]).

In this note, one considers a foliate CICR-submanifold $M$ of a para-Kählerian manifold $\tilde{M}(J,\tilde{\Omega},\tilde{g})$. It is proved that the necessary and sufficient condition in order that the leaf $M^\top$ of the horizontal distribution $D^\top$ on $M$ carries a $J$-skew-symmetric vector field $X$, that is, $\nabla X = X \wedge JX$, is that the vertical distribution $D^\perp$ on $M$ is autoparallel.

In this case, $M$ may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is an invariant totally geodesic submanifold of $M$ and $M^\perp$ is an isotropic totally geodesic submanifold.

Furthermore, if $\Omega$ is the symplectic form of $M^\top$, it is shown that $X$ is a global Hamiltonian of $\Omega$ and $JX$ is a relative infinitesimal automorphism of $\Omega$ (a similar discussion can be made for proper CR-submanifolds of a Kählerian manifold).

2. Preliminaries. Let $\tilde{M}(J,\tilde{\Omega},\tilde{g})$ be a $2m$-dimensional para-Kählerian manifold, where, as is well known [7], the triple $(J,\tilde{\Omega},\tilde{g})$ of tensor fields is the paracomplex operator, the symplectic form, and the para-Hermitian metric tensor field, respectively.

If $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, these manifolds satisfy

$$J^2 = \text{Id}, \quad d\tilde{\Omega} = 0, \quad (\tilde{\nabla}J)\tilde{Z} = 0, \quad \tilde{Z} \in \Gamma T\tilde{M}. \quad (2.1)$$

Let $x : M \to \tilde{M}$ be the immersion of an $l$-codimensional submanifold $M$, $l < m$, in $\tilde{M}$ and let $T_p^\perp M$ and $T_p M$ be the normal space and the tangent space at each point $p \in M$. 
If \( J(T_p^\perp M) \subset T_pM \), then \( M \) is said to be a *coisotropic* submanifold of \( \tilde{M} \) (see [2]). If \( \tilde{W} = \text{vect}\{h_a, h_a^*; a = 1, \ldots, m, a^* = a + m\} \) is a real Witt vector basis on \( \tilde{M} \), one has

\[
\tilde{g}(h_a, h_b) = \tilde{g}(h_a^*, h_b^*) = \delta_{ab}. \tag{2.2}
\]

Next, if \( \tilde{W}^* = \{\omega^a, \omega^a^*\} \) denotes the associated cobasis of \( \tilde{W} \), then \( \tilde{g} \) and \( \tilde{\Omega} \) are expressed by

\[
\tilde{g} = 2 \sum \omega^a \otimes \omega^a^*, \tag{2.3}
\]
\[
\tilde{\Omega} = \sum \omega^a \wedge \omega^a^*. \tag{2.4}
\]

We recall also that \( \tilde{W} \) may split as

\[
\tilde{W} = \tilde{S} + \tilde{S}^*, \tag{2.5}
\]

where the pairing \((\tilde{S}, \tilde{S}^*)\) defines an involutive automorphism of square 1, that is,

\[
Jh_a = h_a^*, \quad Jh_a^* = h_a, \tag{2.6}
\]

and the local connection forms \( \tilde{\theta}^A_B \in \Lambda^1 \tilde{M}, A, B \in \{1, 2, \ldots, 2m\} \) satisfy

\[
\tilde{\theta}^a_{B} = 0, \quad \tilde{\theta}^a_{B^*} = 0, \quad \tilde{\theta}^a_{B} + \tilde{\theta}^a_{B^*} = 0. \tag{2.7}
\]

It has been proved in [10] that any coisotropic submanifold \( M \) of a para-Kählerian manifold \( \tilde{M} \) is a CR-submanifold of \( \tilde{M} \) and such a submanifold has been called a CICR-submanifold [6].

Let \( D^\perp : p \rightarrow D_p^\perp = T_pM \setminus J(T_p^\perp M) \) and \( D^\perp : p \rightarrow D_p^\perp = J(T_p^\perp M) \subset T_pM \) be the two complementary differentiable distributions on \( M \). One has

\[
JD^\perp_p = D_p^\perp, \quad JD^\perp_p = T_p^\perp M, \tag{2.8}
\]

and \( D^\perp \) (resp., \( D^\perp \)) is called the *horizontal* (resp., *vertical*) distribution on \( M \).

As in the Kählerian case, the vertical distribution \( D^\perp \) is always involutive.

If \( M \) is defined by the Pfaffian system

\[
\omega^r = 0, \quad r = 2m + 1 - l, \ldots, 2m, \tag{2.9}
\]

then one has

\[
D_p^\perp = \{h_i, h_i^*; i = 1, \ldots, m - l, i^* = i + m\}, \tag{2.10}
\]
\[
D_p^\perp = \{h_r, r = m - l + 1, \ldots, m\}.
\]

Further denote by

\[
\varphi^\perp = \omega^{m-l+1} \wedge \cdots \wedge \omega^m \tag{2.11}
\]

the simple unit form which corresponds to \( D^\perp \).
Then, in order that the distribution $D^\top$ be also involutive, it is necessary and sufficient that $\varphi^\perp$ be a conformal integral invariant of $D^\top$, that is,

$$\mathcal{L}_{D^\top} \varphi^\perp = f \varphi^\perp$$  \hspace{1cm} (2.12)

for a certain scalar function $f$.

By a standard calculation, one derives that the above equation implies

$$\theta^r_i = 0,$$  \hspace{1cm} (2.13)

and in this case, one may write

$$d\varphi^\perp = -\left(\sum \theta^r_i\right) \wedge \varphi^\perp,$$  \hspace{1cm} (2.14)

that is, $\varphi^\perp$ is exterior recurrent.

In this case, as is known [2, 10], $M$ is a foliated CR-submanifold of $\tilde{M}$.

We will investigate now the case when the leaf $M^\top$ of $D^\top$ carries a $J$-skew-symmetric vector field $X$, that is,

$$\nabla X = X \wedge JX.$$  \hspace{1cm} (2.15)

One may express $\nabla X$ as

$$\nabla X = (JX)^\flat \otimes X - X^\flat \otimes JX,$$  \hspace{1cm} (2.16)

where

$$X = X^ih_i + X^i* h_i* = X^i \omega^* + X^i* \omega^i.$$  \hspace{1cm} (2.17)

Recalling Cartan structure equations [4],

$$\nabla h = \theta \otimes e \in A^1(M, TM),$$

$$d\omega = -\theta \wedge \omega,$$

$$d\theta = -\theta \wedge \theta + \Theta.$$  \hspace{1cm} (2.18)

In the above equations, $\theta$, respectively $\Theta$, are the local connection forms in the bundle $W$, respectively the curvature forms on $M$.

Then making use of Cartan structure equations, one finds by a standard calculation that (2.16) implies that the vertical distribution $D^\perp$ is autoparallel, that is, $\nabla_{Z'} Z'' \in D^\perp$, for all $Z', Z'' \in D^\perp$, which, in terms of connection forms, is expressed by

$$\theta^i_r = 0.$$  \hspace{1cm} (2.19)

We agree to call $\theta^i_r$ and $\theta^r_i$ the mixed connection forms.

Taking account of (2.13) and (2.19), one derives from (2.16)

$$dX^b = 2(JX)^b \wedge X^b,$$  \hspace{1cm} (2.20)

which agrees with the general equation of skew-symmetric killing vector fields [5, 8].
Next, by (2.1), one has
\[
\nabla JX = (JX)^b \otimes JX - X^b \otimes X, \tag{2.21}
\]
which shows that $JX$ is a gradient vector field.

Hence, we may state the following theorem.

**Theorem 2.1.** Let $x : M \to \tilde{M}$ be an improper immersion of a CR-submanifold in a para-Kählerian manifold $\tilde{M}(J, \tilde{\Omega}, \tilde{g})$ and let $D^\top$ (resp., $D^\perp$) be the horizontal distribution (resp., the vertical distribution) on $M$. If $M$ is a foliate CR-submanifold, then the necessary and sufficient condition in order that the leaf $M^\top$ of $D^\top$ carries a $J$-skew-symmetric vector field $X$ is that $D^\perp$ is an autoparallel foliation. In this case, the CR-submanifold $M$ under consideration may be viewed as the local Riemannian product $M = M^\top \times M^\perp$, where $M^\top$ is an invariant totally geodesic submanifold of $M$ and $M^\perp$ is an isotropic totally geodesic submanifold. In addition, in this case, $JX$ is a gradient vector field.

**3. Properties.** In this section, we will point out some additional properties of $X$ involving the symplectic form $\Omega$ of $M^\top$ and the exterior covariant differential $d^V$ of $\nabla X$.

Operating on (2.16) and (2.21), one derives by a short calculation
\[
\begin{align*}
d^V(\nabla X) &= \nabla^2 X = 2(X^b \wedge (JX)^b) \otimes JX, \\
d^V(\nabla JX) &= \nabla^2 JX = 2(X^b \wedge (JX)^b) \otimes X,
\end{align*}
\]
which gives
\[
\begin{align*}
\nabla^2(X + JX) &= 2(X^b \wedge (JX)^b) \otimes (X + JX), \\
\nabla^2(X - JX) &= -2(X^b \wedge (JX)^b) \otimes (X - JX).
\end{align*}
\]

Therefore, we agree to define $X + JX$ and $X - JX$ as $2$-covariant recurrent vector fields.

It should also be noticed that by reference to the general formula
\[
\nabla_X (X_1 \wedge \cdots \wedge X_p) = \sum (X_1 \wedge \cdots \wedge \nabla_X X_j \wedge \cdots \wedge X_p), \quad V \in \Gamma TM, \tag{3.3}
\]
one finds by (2.15) and (2.21)
\[
\nabla_V (X \wedge JX) = 2g(V, JX)(X \wedge JX). \tag{3.4}
\]

This shows that the covariant derivative of $X \wedge JX$ with respect to any vector field $V$ is proportional to $X \wedge JX$.

On the other hand, by the general formula
\[
\nabla^2 V(Z, Z') = R(Z, Z')V, \tag{3.5}
\]
where $R$ denotes the curvature tensor field and $V, Z, Z'$ are vector fields, one has (see also [9])

$$\mathcal{R}(Z,V) = \text{Tr} R(\cdot,Z)V,$$

(3.6)

where $\mathcal{R}$ is the Ricci tensor field of $\nabla$.

Since in the case under consideration one must take in (3.6) the para-Hermitian trace, then setting in (3.6) $Z = V = X$, one finds

$$\mathcal{R}(X,X) = 0,$$

(3.7)

that is, the Ricci curvature of $X$ vanishes.

Denote by $\tilde{\Omega}$ the symplectic form of $\tilde{M}$, then $\Omega = \tilde{\Omega}|_{M^\top}$ is a symplectic form of rank equal to the dimension of $M^\top$, that is, in our case, $2(m - l)$.

Then, if $\flat Z : Z \rightarrow -i_Z \Omega$ is the symplectic isomorphism, by a short calculation and on behalf of (2.4), one gets

$$\flat X = -(jX)^\flat,$$

(3.8)

and since $jX$ is a gradient vector field, we conclude according to a known definition (see also [1]) that $X$ is a global Hamiltonian of $\Omega$.

In a similar manner, one finds

$$\flat (jX) = X^\flat,$$

(3.9)

and by (2.20), it follows that

$$\delta(\mathcal{L}_{jX}\Omega) = 0,$$

(3.10)

which shows that $jX$ is a relative infinitesimal automorphism of $\Omega$ [1].

We state the following theorem.

**Theorem 3.1.** Let $M$ be a CR-submanifold of a para-Kählerian manifold $\tilde{M}$ and let $\Omega$ be the symplectic form on $M^\top$. If $M$ carries a $J$-skew-symmetric vector field $X$, then the following properties hold:

(i) $X$ is a global Hamiltonian of $\Omega$ and $jX$ is a relative infinitesimal automorphism of $\Omega$;

(ii) the Ricci tensor field $\mathcal{R}(X,X)$ vanishes;

(iii) the vector fields $X + jX$ and $X - jX$ are $2$-covariant recurrent.

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