THE EIGENVALUE PROBLEM FOR THE \( p \)-LAPLACIAN-LIKE EQUATIONS

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We consider the eigenvalue problem for the following \( p \)-Laplacian-like equation:

\[
-\text{div} \left( a \left( |Du|^p \right) |Du|^{p-2}Du \right) = \lambda f(x,u) \quad \text{in} \quad \Omega,
\]

\[
u(x) = 0 \quad \text{on} \quad \partial \Omega,
\]

\( \Omega \subset \mathbb{R}^n \) is a bounded smooth domain. When \( \lambda \) is small enough, a multiplicity result for eigenfunctions are obtained. Two examples from nonlinear quantized mechanics and capillary phenomena, respectively, are given for applications of the theorems.

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1. Introduction. This paper is devoted to the study of the eigenvalue problem for the \( p \)-Laplacian-like equation

\[
-\text{div} \left( a \left( |Du|^p \right) |Du|^{p-2}Du \right) = \lambda f(x,u), \quad x \in \Omega,
\]

\[
u(x) = 0, \quad x \in \partial \Omega,
\]

\( \lambda > 0 \) is a real parameter, \( 1 < p < n \), \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \), and \( Du \) denotes the gradient of \( u \), \( f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \), \( a \in C(\mathbb{R}^+, \mathbb{R}) \).

We call \( \lambda \) an eigenvalue of (1.1) provided (1.1), for this \( \lambda \), has a nontrivial weak solution, say \( u_\lambda \), which is then called an eigenfunction corresponding to \( \lambda \). Denote

\[
A(r) = \int_0^r a(s) \, ds, \quad F(x,t) = \int_0^t f(x,s) \, ds.
\]

We look for nontrivial solutions of (1.1), and this question is reduced to show, for some \( \lambda \in \mathbb{R} \), the existence of critical points for the functional

\[
I_\lambda(u) = \frac{1}{p} \int_\Omega A(|Du|^p) \, dx - \lambda \int_\Omega F(x,u) \, dx, \quad u \in E = W_{0}^{1,p}(\Omega).
\]

In [5], Pielichowski discussed the existence and nonnegativity of the first eigenvalue and eigenfunction, in a weak sense, of the \( p \)-Laplace equations with some kind of nonlinear terms below

\[
-\text{div}(|Du|^{p-2}Du) + a(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u.
\]
Under the assumption that \( A(\sigma) \leq \phi(\sigma(x))/|\psi(\sigma(x))| \leq B(\sigma) \) where \( A(\sigma), B(\sigma) \) are constants, Garcia-Huidobro et al. [4] proved the existence of eigenvalues and eigenfunctions for the \( p \)-Laplacian-like equation in the radial form

\[
[r^{n-1}\phi(u')]' + \lambda r^{n-1}\psi(u) = 0, \quad r \in (0,R), \\
u'(0) = 0, \quad u(R) = 0.
\] (1.5)

They used the fixed-point theorem and continuation to techniques. Recently, Boccardo [2] showed the existence of positive eigenfunctions to a kind of \( p \)-Laplace-like equations

\[
-\text{div}(M(x,u)Du) = \lambda u, \quad x \in \Omega, \\
u > 0, \quad x \in \Omega, \\
\|u\|_{L^2(\Omega)} = r \quad r \in \mathbb{R}^+.
\] (1.6)

We are especially interested in Ubilla’s paper [7], which studied the solvability of the boundary value problem for \( p \)-Laplacian-like equation in the radial form

\[
-(a(|u'(r)|^p)|u'(r)|^{p-2}u'(r))' = f(u(r)) \quad r \in I = (0,1) \\
u(0) = u(1) = 0.
\] (1.7)

Under the assumption that

\[
a(|t|^p)|t|^{p-2}t \in C^1(R\setminus\{0\},R) \cap C(R,R), \quad (a(|t|^p)|t|^{p-2}t)' > 0, \quad \forall t \neq 0,
\] (1.8)

a multiplicity result was obtained by using energy relations and the shooting method. The key of our trick is to change this assumption into that the mapping \( r \mapsto A(|r|^p) \) defined in (1.2) is strictly convex, and then consider the eigenvalue problem (1.1). Also, the method we used, the mountain pass theorem and the minimax principle, is different from [7] and some other related papers (see [7] and the references therein). We got the existence of two eigenfunctions \( u_\lambda, v_\lambda \) not necessarily radial ones. In addition, we found that the behaviors of these two eigenfunctions near \( \lambda = 0 \) are much different as \( \lim_{\lambda \to 0^+} \|u_\lambda\|_E = +\infty, \lim_{\lambda \to 0^+} \|v_\lambda\|_E = 0 \). Our idea comes partially from [1].

2. Main results. Assume that
(A1) the mapping \( r \mapsto B(r) = A(|r|^p) \) is strongly convex;
(A2) there exist constants \( c_0 > 0, T > 0 \) such that \( A(t) \geq c_0 t \), for all \( t \geq 0 \) and \( a(s) \leq T, \) for all \( s \geq 0 \);
(A3) there exist constants \( b_0 > 0, b_1 > 0 \) such that for all \( x \in \Omega, \\
|f(x,u)| \leq b_0 |u|^r - 1 + b_1 |u|^q - 1, \quad \text{for} \ 1 < q < r < p^*, \quad p^* = \frac{np}{n-p}; \) (2.1)
(A4) there exist constants $t_0, \theta$ such that $0 < \theta < c_0/pT$ where $c_0, T$ are constants as in (A2), and

$$\theta f(x,t)t > F(x,t) > 0, \quad \forall x \in \overline{\Omega}, \ 0 < t_0 < |t|; \quad (2.2)$$

(A5) for all $x \in \overline{\Omega}$, $t \geq 0$, $f(x,t) \geq 0$, it holds that

$$\lim_{t \rightarrow 0^+} \frac{F(x,t)}{t^p} = +\infty. \quad (2.3)$$

Then we have the main results.

**Theorem 2.1.** Under assumptions (A1) to (A5), there exists a number $\lambda^* > 0$ such that for each $\lambda \in (0,\lambda^*)$, there exists an eigenfunction $u_\lambda$ of (1.1) satisfying $\lim_{\lambda \rightarrow 0} \|u_\lambda\|_E = +\infty$.

**Theorem 2.2.** Assume (A1) to (A5) and $f(x,t) \geq 0$, then there is a number $\lambda^* > 0$ such that for each $\lambda \in (0,\lambda^*)$, (1.1) has one eigenfunction $u_\lambda$ behaving $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_E = 0$.

3. Proof of the main results

**Lemma 3.1.** Assume (A1) to (A4), then $I_\lambda$ defined in (1.3) belongs to $C^1(E,R)$.

**Proof.** Denote

$$I_A(u) = \frac{1}{p} \int_{\Omega} A(|Du|^p) \, dx, \quad I_F(u) = \lambda \int_{\Omega} F(x,u) \, dx, \quad u \in E \quad (3.1)$$

so $I_\lambda(u) = I_A(u) - I_F(u)$. We will then complete the proof by the following two claims.

**Claim 1 ($I_\lambda \in C^1(E,R)$).** In fact, by (A1), for all $\lambda \in (0,1)$, $\varphi \in E$, we have

$$\int_0^{\lambda|Du| + (1-\lambda)(|Du| + |D\varphi|)} a(s) \, ds \leq \lambda \int_0^{|Du|^p} a(s) \, ds + (1-\lambda) \int_0^{|Du| + |D\varphi|} a(s) \, ds, \quad (3.2)$$

that is,

$$\int_0^{\lambda|Du + (1-\lambda)D\varphi|} a(s) \, ds - \int_0^{|Du|^p} a(s) \, ds \leq (1-\lambda) \left( \int_0^{|Du| + |D\varphi|} a(s) \, ds - \int_0^{|Du|^p} a(s) \, ds \right). \quad (3.3)$$
Set, in the above inequality, $1 - \lambda = t$, we then have

$$\frac{I_A(u + t\varphi) - I_A(u)}{t} = \frac{1}{tp} \int_{\Omega} \left( \int_0^{(|Du + tD\varphi|)^p} a(s) \, ds - \int_0^{(|Du|^p)} a(s) \, ds \right) \, dx$$

$$\leq \frac{1}{p} \int_{\Omega} \left( \int_0^{(|Du + tD\varphi|)^p} a(s) \, ds - \int_0^{(|Du|^p)} a(s) \, ds \right) \, dx \quad (3.4)$$

$$< +\infty,$$

which is independent of $t$. Hence, we can apply the Lebesgue dominated convergence theorem to the equality

$$\frac{I_A(u + t\varphi) - I_A(u)}{t} = \frac{1}{p} \int_{\Omega} a(|Du|^p + \eta(|Du + tD\varphi|^p - |Du|^p))$$

$$\cdot \frac{1}{t}(|Du + tD\varphi|^p - |Du|^p) \, dx, \quad \text{for some } \eta \in (0, 1),$$

and letting $t \to 0$, we then get

$$I'_A(u)\varphi = \int_{\Omega} a(|Du|^p)|Du|^{p-2} \cdot D\varphi \, dx. \quad (3.6)$$

Next, we show that $I'_A$ is continuous in $u$. In the following, the constant $C$ may vary line by line.

Suppose $\{u_m\} \subset E$ satisfying $\|u_m - u\|_E \to 0$ as $m \to \infty$. We then claim that $\|I'_A(u_m) - I'_A(u)\| \to 0$. In fact,

$$\|I'_A(u_m) - I'_A(u)\|$$

$$= \sup_{\varphi \in E} \frac{\int_{\Omega} \left( a(|Du_m|^p)|Du_m|^{p-2}Du_m \cdot D\varphi - a(|Du|^p)|Du|^{p-2}Du \cdot D\varphi \right) \, dx}{\|\varphi\|_E}$$

$$\leq \frac{1}{p} \|B'(Du_m) - B'(Du)\|_{L^{p'}(\Omega)}, \quad (3.7)$$

where

$$B'(r) \equiv DB(r) = pa(|r|^p)|r|^{p-2}r, \quad r \in \mathbb{R}^n, \quad p' = \frac{p}{p-1}. \quad (3.8)$$

Because $u_m \to u$ in $E$, by Egorov theorem, for any $\eta > 0$ there exists $\Omega_\eta \subset \Omega$ such that $|\Omega \setminus \Omega_\eta| < \eta$ and $u_m, Du_m$ converge uniformly to $u, Du$, respectively,
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in $\Omega_\eta$. Also, $\Omega_\eta$ can be chosen large enough so that the following holds as well

$$\int_{\Omega \setminus \Omega_\eta} |Du|^p \, dx < \frac{\varepsilon}{2} \quad (3.9)$$

for any given $\varepsilon > 0$. By virtue of $Du_m \to Du$ in $L^p(\Omega)$, when $m$ is large enough,

$$\int_{\Omega \setminus \Omega_\eta} |Du_m|^p \, dx < C \left( \int_{\Omega \setminus \Omega_\eta} |Du_m - Du|^p \, dx + \int_{\Omega \setminus \Omega_\eta} |Du|^p \, dx \right)$$

$$< C \left( \int_{\Omega} |Du_m - Du|^p \, dx + \varepsilon/2 \right) < C\varepsilon. \quad (3.10)$$

Then, by (A2), (3.8), and (3.10), when $m$ is large enough, we obtain

$$\left( \int_{\Omega \setminus \Omega_\eta} |B'(Du_m)|^{p/(p-1)} \, dx \right)^{(p-1)/p} \leq p \left( \int_{\Omega \setminus \Omega_\eta} \left( T |Du_m|^{p-1} \right)^{p/(p-1)} \, dx \right)^{(p-1)/p} \leq T(C\varepsilon)^{(p-1)/p},$$

that is, $\|B'(Du_m)\|_{L^{p'}(\Omega \setminus \Omega_\eta)} \leq C\varepsilon$. Similarly,

$$\|B'(Du)\|_{L^{p'}(\Omega \setminus \Omega_\eta)} \leq C\varepsilon. \quad (3.12)$$

Noticing that

$$\|B'(Du_m) - B'(Du)\|_{L^{p'}(\Omega \setminus \Omega_\eta)} \leq \|B'(Du_m) - B'(Du)\|_{L^{p'}(\Omega \setminus \Omega_\eta)}$$

$$+ \|B'(Du_m)\|_{L^{p'}(\Omega \setminus \Omega_\eta)} + \|B'(Du)\|_{L^{p'}(\Omega \setminus \Omega_\eta)},$$

(3.13)

We then get $\|I'_A(u_m) - I'_A(u)\| \to 0$ as $m \to \infty$. Therefore, $I'_A$ is continuous at the point $u$, that is, $I'_A \in C^1(E,R)$. 

**Claim 2** ($I_F \in C^1(E,R)$). The proof is similar to Claim 1 and we then omit it. This completes the proof of Lemma 3.1.

**Lemma 3.2.** Assume (A1) to (A4), then $I_\lambda$ satisfies (PS) condition.

**Proof.** From Lemma 3.1, we know that

$$I'_\lambda(u)\varphi = \int_{\Omega} \left[ a(|Du|^p) |Du|^{p-2} Du \cdot D\varphi - \lambda f(x,u) \varphi \right] \, dx \quad \forall u, \varphi \in E. \quad (3.14)$$
Suppose that $S = \{u_m\} \subset E$ satisfies that for some $M > 0$,

$$I_\lambda(u_m) \leq M, \quad \forall u_m \in S, \quad (3.15)$$

$$I_\lambda'(u_m) \to 0. \quad (3.16)$$

We prove below that there exists a subsequence of $\{u_m\}$ converging strongly in $E$.

(a) At first, we show that $S$ is bounded in $E$. From (3.16), for all $\varphi \in E$, it holds that

$$\int_\Omega \left[ a\left( |Du_m|^p \right) |Du_m|^{p-2}Du_m \cdot D\varphi - \lambda f(x, u_m) \varphi \right] dx = o(1) \|\varphi\|_E. \quad (3.17)$$

Using (A4) and (A2), we have

$$I_\lambda(u_m) - \theta I_\lambda'(u_m) u_m = \frac{1}{p} \int_\Omega A\left( |Du_m|^p \right) dx - \theta \int_\Omega a\left( |Du_m|^p \right) |Du_m|^p dx$$

$$+ \lambda \int_\Omega \left[ \theta f(x, u_m) u_m - F(x, u_m) \right] dx$$

$$> \frac{1}{p} \int_\Omega A\left( |Du_m|^p \right) dx - \theta \int_\Omega a\left( |Du_m|^p \right) |Du_m|^p dx$$

$$> \frac{c_0}{p} \int_\Omega |Du_m|^p dx - \theta \int_\Omega T |Du_m|^p dx. \quad (3.18)$$

Combining this with (3.17) yields

$$\left( \frac{c_0}{p} - \theta T \right) \int_\Omega |Du_m|^p dx < M + o(1) \theta \|u_m\|_E, \quad (3.19)$$

which implies

$$\|u_m\|_E \leq C. \quad (3.20)$$

Hence, there exists a subsequence of $S$, still denoted by $\{u_m\}$, such that $u_m \rightharpoonup u$ in $E$ and hence $Du_m \rightharpoonup Du$ in $L^p(\Omega)$, $u_m \to u$ in $L^2(\Omega)$, $1 < s < p^*$.

(b) Set

$$p_m(x) \equiv \left( a\left( |Du_m|^p \right) |Du_m|^{p-2}Du_m - a(|Du|^p) |Du|^{p-2}Du \right) (Du_m - Du), \quad (3.21)$$
then

\[ I_m \equiv \int_\Omega p_m(x) \, dx \]

\[ = \int_\Omega a \left( |Du_m|^p \right) |Du_m|^{p-2} Du_m(Du_m - Du) \, dx \]

\[ - \int_\Omega a(|Du|^p)|Du|^{p-2} Du(Du_m - Du) \, dx \]

\[ \equiv I_m^{(1)} + I_m^{(2)}. \]  

(3.22)

We show below that \( p_m(x) \to 0 \) a.e. in \( \Omega \). As \( Du_m - Du \) in \( L^p(\Omega) \), it is obvious that \( I_m^{(2)} \to 0 \). We choose in (3.17) \( \varphi = u_m - u \), then

\[ I_m^{(1)} = \lambda \int_\Omega f(x,u_m)(u_m - u) \, dx + o(1)\|u_m - u\|_E. \]  

(3.23)

By (A3) and the Sobolev imbedding theorem,

\[ \left| \int_\Omega f(x,u_m)(u_m - u) \, dx \right| \leq \|f(x,u_m)\|_{r'}\|u_m - u\|_r, \quad r' = r/r - 1 \]

\[ \leq (b_0\|u_m^{r-1}\|_{r'} + b_1\|u_m^{r-1}\|_{r'})\|u_m - u\|_r \]

\[ \leq c \left( \|u_m\|_E^{r-1} + \|u_m\|_E^{q-1} \right)\|u_m - u\|_r \]

\[ \to 0, \quad \text{as } m \to \infty. \]  

(3.24)

Therefore, from (3.23), \( I_m^{(1)} \to 0 \) and so \( I_m \to 0 \) as \( m \to \infty \). Because \( B(r) \) is strictly convex, then for all \( r_1, r_2 \in \mathbb{R}^n \), it holds that

\[ (B'(r_1) - B'(r_2)) \cdot (r_1 - r_2) \geq 0, \quad \text{(3.25)} \]

where the equality sign holds if and only if \( r_1 = r_2 \). From this and the definition of \( p_m(x) \), we then get \( p_m(x) \geq 0 \), which with \( I_m \to 0 \) gives \( p_m(x) \to 0 \), a.e. \( x \in \Omega \). So we can find \( \Omega_0 \subset \Omega \) such that \( \text{meas}(\Omega \setminus \Omega_0) = 0 \), \( u_m(x) \to u(x) \) and \( p_m(x) \to 0 \) on \( \Omega_0 \).

(c) Based on (3.25) and the fact that \( p_m(x) \geq 0 \), very similar to the first part of the proof of [3, Lemma 1], we can get \( Du_m(x) \to Du(x) \), for all \( x \in \Omega_0 \).

(d) At last, we prove \( \|u_m - u\|_E \to 0 \). From the step (c), \( Du_m \to Du \), a.e. \( x \in \Omega \). By Egorov theorem, for any \( \delta > 0 \), there exists \( \Omega_\delta \subset \Omega \) such that \( \text{meas}(\Omega \setminus \Omega_\delta) < \delta \) and \( Du_m \) converges uniformly to \( Du \) on \( \Omega_\delta \). Because \( B(r) \) is convex, then for any \( r_1, r_2 \in \mathbb{R}^n \) we have

\[ B'(r_1) \cdot (r_1 - r_2) \geq B(r_1) - B(r_2). \]  

(3.26)
Choosing $r_2 = 0$, with $B(0) = A(0) = 0$, then

$$B'(r_1) \cdot r_1 \geq B(r_1) = A(|r_1|^p) \geq c_0 |r_1|^p. \quad (3.27)$$

Suppose $\Omega' \subset \Omega$, by (3.27) and (3.8). Using (A2) and Young’s inequality, we get

$$\frac{c_0}{p} \int_{\Omega'} |Du_m(x)|^p \, dx \leq \int_{\Omega'} a(|Du_m|^p) |Du_m|^p \, dx$$

$$= \int_{\Omega'} p_m(x) \, dx + \int_{\Omega'} a(|Du_m|^p) |Du_m|^{p-2} Du_m \cdot Du \, dx$$

$$+ \int_{\Omega'} a(|Du|^p) |Du|^{p-2} Du \cdot Du_m \, dx$$

$$- \int_{\Omega'} a(|Du|^p) |Du|^p \, dx$$

$$\leq \int_{\Omega'} p_m(x) \, dx + T \int_{\Omega'} |Du_m|^{p-1} |Du| \, dx$$

$$+ T \int_{\Omega'} |Du|^{p-1} |Du_m| \, dx + T \int_{\Omega'} |Du|^p \, dx$$

$$\leq \int_{\Omega'} p_m(x) \, dx + \varepsilon_1 \int_{\Omega'} |Du_m|^p \, dx + C(\varepsilon_1) \int_{\Omega'} |Du|^p \, dx$$

$$+ \varepsilon_2 \int_{\Omega'} |Du_m|^p \, dx + C(\varepsilon_2) \int_{\Omega'} |Du|^p \, dx$$

$$+ T \int_{\Omega'} |Du|^p \, dx. \quad (3.28)$$

Setting $\varepsilon_1 = \varepsilon_2 = c_0/4p$ in the above inequality yields

$$\frac{c_0}{2p} \int_{\Omega'} |Du_m(x)|^p \, dx \leq \int_{\Omega'} p_m(x) \, dx + C \int_{\Omega'} |Du|^p \, dx. \quad (3.29)$$

Let $|\Omega'|$ be small enough so that for a given $\varepsilon > 0$ there holds

$$\int_{\Omega'} |Du|^p \, dx < \varepsilon. \quad (3.30)$$

Since $I_m \to 0$ and $p_m(x) > 0$, then when $m$ is large enough we have

$$\int_{\Omega'} p_m(x) \, dx \leq \int_{\Omega} p_m(x) \, dx < \varepsilon. \quad (3.31)$$
Combining this with (3.29), we get \( \int_{\Omega'} |Du_m(x)|^p \, dx < C\varepsilon \) when \( m \) become large enough. Noticing \( Du_m - Du \) uniformly on \( \Omega \setminus \Omega' \), then

\[
\|Du_m - Du\|_{LP(\Omega')} = \|Du_m - Du\|_{LP(\Omega,\Omega')} + \|Du_m - Du\|_{LP(\Omega')'}
\]

\[
\leq \|Du_m - Du\|_{LP(\Omega,\Omega')} + \|Du_m\|_{LP(\Omega')} + \|Du\|_{LP(\Omega')'} \quad (3.32)
\]

\[
\leq C\varepsilon \quad \text{as } m \text{ is large enough.}
\]

This completes the proof of Lemma 3.2. \( \square \)

**Proof of Theorem 2.1.** We complete the proof by three steps.

**Step 1.** In fact, from (A3) we find

\[
|F(x,u)| \leq \frac{b_0}{r} |u|^r + \frac{b_1}{q} |u|^q, \quad x \in \Omega. \quad (3.33)
\]

Condition (A2) and the Sobolev imbedding theorem yield

\[
I_\lambda(u) \geq \frac{c_0}{p} \int_\Omega |Du|^p \, dx - \lambda \int_\Omega \left( \frac{b_0}{r} |u|^r + \frac{b_1}{q} |u|^q \right) \, dx
\]

\[
\geq \frac{c_0}{p} \|u\|_E - k_0 \lambda \|u\|_E^r - k_1 \lambda \|u\|_E^q,
\]

where \( k_0 > 0, k_1 > 0 \) are constants and independent of \( u \).

Suppose \( u \in E \) satisfying that \( \|u\|_E = \lambda^{-\alpha} \), \( 0 < \alpha < 1/(r-p) \), then by (3.34) we have

\[
I_\lambda(u) \geq \frac{c_0}{p} \lambda^{-\alpha p} - k_0 \lambda^{-\alpha r} - k_1 \lambda^{-\alpha q}.
\]

(3.35)

Because \( 0 < \alpha < 1/(r-p) \), then \( \alpha \lambda \equiv (c_0/p)\lambda^{-\alpha p} - k_0 \lambda^{-\alpha r} - k_1 \lambda^{-\alpha q} \rightarrow +\infty \)

as \( \lambda \to 0^+ \). Hence, there exists \( \lambda^* > 0 \) small enough such that \( \alpha \lambda > 0 \) for all \( \lambda \in (0,\lambda^*) \). Then, we get

\[
I_\lambda(u) \geq \alpha \lambda > 0 \quad \text{for } \|u\|_E = \rho_\lambda,
\]

(3.36)

where \( \rho_\lambda = \lambda^{-\alpha} \).

**Step 2.** Condition (A4) implies that

\[
F(x,t) > d_0 t^{1/q} - d_1, \quad \forall (x,t) \in \overline{\Omega} \times R,
\]

(3.37)
where \(d_0, d_1\) are positive constants. Using (3.37) and condition (A2), we find

\[
I_{\lambda}(tv) = \frac{1}{p} \int_{\Omega} A(t^p |Dv|^p) \, dx - \lambda \int_{\Omega} F(x, tv) \, dx \\
\leq \frac{T}{p} \int_{\Omega} t^p |Dv|^p \, dx - \lambda \int_{\Omega} (d_0 t^{1/\theta} v^{1/\theta} - d_1) \, dx \\
= \frac{T}{p} t^p \|v\|_E^p - \lambda d_0 t^{1/\theta} \|v\|_E^{1/\theta} + \lambda d_1. \tag{3.38}
\]

Condition (A2) implies that \(c_0 \leq T\), and then by (A4) we get \(p < \frac{1}{\theta}\). Thus as \(t \to +\infty\), \(I_{\lambda}(tv) \to -\infty\).

**Step 3.** By Lemma 3.2, \(I_{\lambda}\) satisfies the (PS) condition. Then, by the results of Steps 1 and 2, we can apply the mountain pass theorem to get that there exists a nontrivial critical point \(u_{\lambda}\) of \(I_{\lambda}\) such that

\[
I_{\lambda}(u_{\lambda}) = c_{\lambda} \geq \alpha_{\lambda} > 0, \tag{3.39}
\]

and then

\[
I_{\lambda}(u_{\lambda}) \leq \frac{1}{p} \int_{\Omega} T |Du_{\lambda}|^p \, dx + \lambda \int_{\Omega} \left( \frac{b_0}{r} |u_{\lambda}|^r + \frac{b_1}{q} |u_{\lambda}|^q \right) \, dx \\
= \frac{T}{p} \|u_{\lambda}\|_E^p + \frac{\lambda b_0}{r} \|u_{\lambda}\|_r^r + \frac{\lambda b_1}{q} \|u_{\lambda}\|_q^q \tag{3.40}
\]

Let \(\lambda \to 0^+\) in (3.40) as \(\alpha_{\lambda} \to +\infty\), then we obtain \(\|u_{\lambda}\|_E \to +\infty\). This completes the proof.

**Proof of Theorem 2.2.** For \(0 < \alpha < 1/p\), let \(\|u\|_E = \lambda^\alpha\). By (3.34), we have

\[
I_{\lambda}(u) \geq \frac{c_0}{p} \lambda^{\alpha p} - k_0 \lambda^{1+\alpha r} - k_1 \lambda^{1+\alpha q} = \lambda \left( \frac{c_0}{p} \lambda^{\alpha p - 1} - k_0 \lambda^{\alpha r} - k_1 \lambda^{\alpha q} \right). \tag{3.41}
\]

As \(\alpha p - 1 < 0\), then there exists \(\lambda^* > 0\) small enough so that \(I_{\lambda}(u) > 0\) for \(\lambda \in (0, \lambda^*)\), that is,

\[
I_{\lambda}(u) > 0, \quad \forall 0 < \lambda < \lambda^*, \ \|u\|_E = \rho_{\lambda}, \tag{3.42}
\]

where \(\rho_{\lambda} = \lambda^\alpha\). Set \(B_{\rho_{\lambda}} = \{u \in E : \|u\|_E < \rho_{\lambda}\}\), then for \(u \in B_{\rho_{\lambda}}\), by (3.34), we find

\[
I_{\lambda}(u) \geq \frac{c_0}{p} \|u\|_E^p - k_0 \lambda \|u\|_E^r - k_1 \lambda \|u\|_E^q \\
\geq -k_0 \lambda \rho_{\lambda}^r - k_1 \lambda \rho_{\lambda}^q \geq -k_0 (\lambda^*)^{1+r\alpha} - k_1 (\lambda^*)^{1+q\alpha}, \tag{3.43}
\]
then $I_\lambda$ is bounded blow on $\overline{B}_{\rho\lambda}$. Choosing $v \in C_0^\infty(\Omega)$, $0 < v < 1$, $0 \leq |Dv| \leq 1$, $t \geq 0$, then

$$I_\lambda(tv) = \frac{1}{p} \int_\Omega A(t^p|Dv|^p) \, dx - \lambda \int_\Omega F(x,tv) \, dx \leq \frac{T}{p} \int_\Omega t^p|Dv|^p \, dx - \lambda \int_\Omega F(x,tv) \, dx \leq t^p \left[ \frac{T}{p} \int_\Omega |Dv|^p \, dx \inf_{x \in \pi} \frac{F(x,t)}{t^p} \int_\Omega \frac{F(x,tv)}{F(x,t)} \, dx \right].$$

(3.44)

From (A5), we know that $f(x,t) \geq 0$, for all $x \in \overline{\Omega}$, $t \geq 0$ and hence $F(x,tv)/F(x,t) \leq 1$. By (3.44), (A5), and applying the dominated convergence theorem to (3.44), we find that there exist $\delta > 0$, $0 < t < \delta$, $tv \in B_{\rho\lambda}$ such that

$$I_\lambda(tv) < 0. \quad (3.45)$$

Because $I_\lambda$ satisfies the (PS) condition, the minimax theorem on $\overline{B}_{\rho\lambda}$ claims that $I_\lambda$ has a nontrivial critical point $u_\lambda \in B_{\rho\lambda}$, which is a local minimum and $I_\lambda(u_\lambda) < 0$. Then $\|u_\lambda\|_E < \rho\lambda = \lambda^\alpha$, $\|u_\lambda\|_E \to 0$ as $\lambda \to 0^+$. This ends the proof.

4. Examples

**Example 4.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. We consider the $p$-Laplacian problem from nonlinear quantized mechanics as

$$-\text{div}(|Du|^{p-2}Du) = \lambda(|u|^{q-2}u + |u|^{r-2}u), \quad x \in \Omega,$n

$$u(x) = 0, \quad x \in \partial\Omega, \quad (4.1)$$

where $\lambda > 0$, $1 < p < n$, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $1 < q < p < r < p^*$, $p^* = np/(n-p)$. In this case, $a(s) = 1$, $B(r) = |r|^p$ is strictly convex, and conditions (A2) and (A4) are satisfied for $c_0 = T = 1$ while $0 < \theta < 1/p$. Obviously, (A3) also holds. These conditions have been posted directly on the given functions in some papers which dealt with the solvability of the boundary value or eigenvalue problem for the $p$-Laplacian equation (see [6] and the references therein). Then, by virtue of Theorems 2.1 and 2.2, when $\lambda$ is small enough, problem (4.1) possesses at least two eigenfunctions $u_\lambda$ and $v_\lambda$, and

$$\lim_{\lambda \to 0} \|u_\lambda\|_E = +\infty, \quad \lim_{\lambda \to 0} \|v_\lambda\|_E = 0. \quad (4.2)$$
Example 4.2. Consider the eigenvalue problem for generalized capillarity equation originated from the capillary phenomena

\[-\text{div}\left(\left(1 + \frac{|Du|^p}{\sqrt{1 + |Du|^{2p}}}\right)|Du|^{p-2}Du\right) = \lambda(|u|^{q-2}u + |u|^{r-2}u), \quad x \in \Omega,\]

\[u(x) = 0, \quad x \in \partial \Omega,\]

(4.3)

where $\lambda > 0$, $1 < q < p$, $2p < r < p^*$, $p^* = np/(n-p)$. We also can check that (A1) to (A5) are satisfied. By Theorems 2.1 and 2.2, there exist two eigenfunctions $u_\lambda$ and $v_\lambda$ and $\lim_{\lambda \to 0} \|u_\lambda\|_E = +\infty$, $\lim_{\lambda \to 0} \|v_\lambda\|_E = 0$.

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