ON THE STEENROD OPERATIONS IN CYCLIC COHOMOLOGY

MOHAMED ELHAMDADI and YASien GH. GOUDA

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For a commutative Hopf algebra $A$ over $\mathbb{Z}/p$, where $p$ is a prime integer, we define
the Steenrod operations $P^i$ in cyclic cohomology of $A$ using a tensor product of
a free resolution of the symmetric group $S_n$ and the standard resolution of the
algebra $A$ over the cyclic category according to Loday (1992). We also compute
some of these operations.

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1. Introduction. For any prime $p$, the mod $p$ Steenrod algebra $\mathcal{A}(p)$ is the
graded associative algebra generated by the mod $p$ stable operations $P^i$ of de-
gree $2i(p-1)$ in the ordinary cohomology theory. When $p = 2$, it is generated
by the Steenrod squares $Sq^i$ ($i \geq 1$) subject to the Adem relations. The oper-
ations $P^i$ and $Sq^i$ increase degree, respectively, by $2i(p-1)$ and $i$; in other
words,

$$
\begin{align*}
P^i : H^q(-,\mathbb{Z}/p) &\rightarrow H^{q+2i(p-1)}(-,\mathbb{Z}/p), \\
Sq^i : H^q(-,\mathbb{Z}/p) &\rightarrow H^{q+i}(-,\mathbb{Z}/p).
\end{align*}
$$

(1.1)

In [4], Epstein introduced the Steenrod operations into derived functors and
obtained as a special case the Steenrod operations in the cohomology of groups
and in the cohomology of a space with coefficients in sheaves (see also [15]).
Other operations like Adams’ were studied in [5, 11]. The $S$- and $\lambda$-operations
in cyclic homology have been defined and studied in [2]. Some special oper-
ations (dot product, bracket) on Hochschild complex that induce a structure
of graded algebra on the cohomology have been considered in [16]. Steenrod
operations on the Hochschild homology have been studied in [13]. There are
also operations in $K$-theory, for instance [8], and $\lambda$-operations in orthogonal $K$-
theory [3]. Many applications of the Steenrod algebra have been made: in 1958,
Adams [1] used them to compute the stable homotopy groups of spheres and
in the same year Milnor [12] proved that the Steenrod algebra and its dual have
structures of Hopf algebras.

In this paper, we define the Steenrod operations in cyclic cohomology of a
commutative Hopf algebra and obtain some calculations.
2. Steenrod operations on cyclic cohomology. Let $k$ be a commutative ring with unit, $A$ a commutative $k$-Hopf algebra, and $\mathcal{C}$ a cyclic category (see [10, page 202]). We will denote the $k$-algebra over $\mathcal{C}$ by $k[\mathcal{C}]$ and the cyclic category over $A$ by $A^\mathcal{C}$ (see [10]). We define an $A^\mathcal{C}$-structure of cocommutative coalgebra by the formula

$$A^\mathcal{C} \xrightarrow{\bigtriangledown} A \otimes A \xrightarrow{f} A^\mathcal{C} \otimes A^\mathcal{C},$$  \tag{2.1}$$

where $\bigtriangledown$ is $k[\mathcal{C}]$-homomorphism and $f$ is given by

$$f((a_0 \otimes b_0) \otimes (a_1 \otimes b_1) \otimes \cdots \otimes (a_n \otimes b_n)) = (a_0 \otimes a_1 \otimes \cdots \otimes a_n) \otimes (b_0 \otimes b_1 \otimes \cdots \otimes b_n).$$ \tag{2.2}$$

Suppose that $\bigtriangledown^\mathcal{C} = f \circ \bigtriangledown$ gives the cocommutative comultiplication in $A^\mathcal{C} \otimes_k A^\mathcal{C}$, that is, $T \circ \bigtriangledown^\mathcal{C} = \bigtriangledown^\mathcal{C}$, where $T$ is the twisting map $T(a \otimes b) = b \otimes a$. We have, for $x$ in $k[\mathcal{C}]$,

$$f(x[(a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n)]) = x(a_0 \otimes \cdots \otimes a_n) \otimes x(b_0 \otimes \cdots \otimes b_n)$$

$$= x[(a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n)]$$

$$= xf((a_0 \otimes b_0) \otimes \cdots \otimes (a_n \otimes b_n)).$$ \tag{2.3}$$

The comultiplication $\bigtriangledown^\mathcal{C}$ becomes a $k[\mathcal{C}]$-module homomorphism.

2.1. The normalized bar construction. Let $Jk[\mathcal{C}]$ be the cokernel of the $k$-map $k \to k[\mathcal{C}]$. The normalized bar construction of the triple $L = (A^\mathcal{C}, k[\mathcal{C}], k^\mathcal{C})$ is defined to be the graded $k$-module $B(L)$ with

$$B_m(L) = A^\mathcal{C} \otimes_{k[\mathcal{C}]} T^m(Jk[\mathcal{C}]) \otimes_{k[\mathcal{C}]} k^\mathcal{C},$$  \tag{2.4}$$

where $T^m(Jk[\mathcal{C}])$ is the tensor algebra in degree $m$. As $k$-module $B_m(L)$ is spanned by elements written as $a[g_1|\cdots|g_m]u$, where $a$ is in $A^\mathcal{C}$, $g_i$ belongs to $k[\mathcal{C}]$, and $u$ is an element of $k^\mathcal{C}$. The differential $d_m : B_m(L) \to B_{m-1}(L)$ is given by

$$d_m(a[g_1|\cdots|g_m]u) = ag_1[g_2|\cdots|g_m]u$$

$$+ \sum_{i=1}^{m-1} (-1)^i a[g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_m]u$$

$$+ (-1)^m a[g_1|\cdots|g_{m-1}]g_m u.$$ \tag{2.5}$$

The elements are normalized in the sense that $f([g_1|\cdots|g_m]u) = 0$ and $f(a[\cdot]u) = 0$, where $a[\cdot]u$ are elements of $B_0$.

We define also, for the triple $T = (k[\mathcal{C}], k[\Delta\mathcal{C}], k^\mathcal{C})$, the maps $d$ and $f$ in the same manner. Note that for $T$, the differential $d$ is a left $k[\mathcal{C}]$-module homomorphism and $ds + sd = 1 - \sigma f$, where the morphisms $\sigma : k^\mathcal{C} \to B(T)$
and \( s : B_m(T) \to B_{m-1}(T) \) are given by the formulas \( \sigma(u) = [\cdot] u \otimes_k [\cdot] \) and \( s(g[g_1| \cdots |g_m]u) = g[g_1| \cdots |g_m]u \). It is clear that the differential \( d \) in the complex \( B(L) \) is equal to \( 1 \otimes_{k[\langle \epsilon \rangle]} d \). We have the equality

\[
\text{Hom}_{k[\langle \epsilon \rangle]}(B(T), (A^\epsilon)^*) = (B(L))^* = \text{Hom}_{k[\langle \epsilon \rangle]}(B(A^\epsilon), k[\epsilon], k[\epsilon]^\epsilon, (k)^*).
\]

(2.6)

We then have (see [10, page 214]),

\[
HC^n(A) = \text{Ext}^n_{k[\langle \epsilon \rangle]}(A^\epsilon, (k^\epsilon)^*) = H^n((B(L))^*).
\]

(2.7)

Given a triple \( L \) and considering the product \( \sqcup : B(L \otimes L) \to B(L) \otimes B(L) \), we define on \( B(L) \) a structure of coassociative coalgebra by means of comultiplication \( \hat{\Delta} = \sqcup B(\nabla^\epsilon, \nabla_{k[\langle \epsilon \rangle]}, \nabla_{k^\epsilon}) : B(L \otimes L) \to B(L) \otimes B(L) \) and on \( B(L)^* \) the following multiplication as a composite map:

\[
B(L)^* \otimes B(L)^* \to (B(L) \otimes B(L))^* \xrightarrow{(\hat{\Delta})^*} B(L)^*.
\]

(2.8)

We have the following lemma which can be easily proved by ordinary techniques of homological algebra (see [15]).

**Lemma 2.1.** Let \( \mu \) be an arbitrary subgroup of the symmetric group \( S_n \) and \( W \) the free resolution of \( k \) as \( k[\mu] \)-module with a generator \( e_0 \). Then there is a graded \( k[\mu] \)-complex with the following properties:

(a) \( \Delta(w \otimes b) = 0 \) for \( b \in B_0(L) \) and \( w \in W_i, i > 0 \);

(b) \( \Delta(e_0 \otimes b) = \hat{\Delta}_{\otimes r}(b) \) for \( b \in B(L) \) and \( \hat{\Delta}_{\otimes r} : B(L) \to B(L)^{\otimes r} \);

(c) the map \( \Delta : B(L) \to B(L)^{\otimes r} \) is a left \( k[\epsilon] \)-module, homomorphism, where \( k[\epsilon] \) acts on \( W \otimes B(L) \) by \( u(w \otimes b) = w \otimes ub \);

(d) \( \Delta(W_i \otimes B_m(L)) = 0 \) when \( i > (r-1)m \).

Furthermore, there exists a \( k[\mu] \)-homotopy between any two homomorphisms \( \Delta \) with the same properties.

Now define a \( k[\mu] \)-homomorphism \( \theta : W \otimes ((B(L))^*)^{\otimes r} \to (B(L))^* \) with \( \theta(w \otimes x)(m) = B(x) \Delta(w \otimes x), w \in W, x \in ((B(L))^*)^{\otimes r}, m \in B(L) \), and \( B : ((B(L))^*)^{\otimes r} \to ((B(L))^*)^{\otimes r} \) a trivial homomorphism.

**2.2. Operations.** In the above lemma, let \( \mu = \mathbb{Z}/p \) and \( k = \mathbb{Z}/p \), where \( p \) is a prime integer. Consider the \( k[\mathbb{Z}/p] \)-free resolution \( W \) with \( W_i, i \geq 0 \), generated by \( e_i \). For \( i < 0 \), consider \( W_i := W_{-i} \) as a free \( k[\mathbb{Z}/p] \)-module with a generator \( e_{-i} \). Now we define, for \( i \geq 0 \), the homomorphism

\[
R_i : H^q(B(L)^*) \to H^{pq-i}(B(L)^*),
\]

\[
\chi \mapsto R_i(\chi) = \theta^*(e_{-i} \otimes \chi^p).
\]

(2.9)

We extend the definition of this homomorphism to the negative \( i \) by \( R_i = 0 \). The Steenrod operations \( P^i \) are defined in terms of the \( R_j \) in the following manner.
Steenrod maps are the homomorphisms 

efficiency reflexive homology.

\[ L \rightarrow T \rightarrow \text{category} \]

\( \text{and} \)

\( \text{algebra over} \)

\( n< \)

\[ 2.2 \]

\text{Definition}

Consider the triple \( \beta \gamma P_m P_n \)

\[ \beta \gamma P_m P_n = \sum_{i=0}^{n} P_i \otimes P_{n-i} \text{ and } \beta P^n = \sum_{i=0}^{n} \beta P_i \otimes P_{n-i} + P_i \otimes \beta P_{n-i}. \]

\( \text{The Steenrod maps satisfy} \)

\[ p \geq 2 \text{ and } m < p n, \]

\[ \beta y p^m p^n = \sum_i (-1)^{m+i} \left( m - pi + (p - 1)(n - m + i - 1) \right) \beta y p^{m+n-i} p^i, \quad (2.10) \]

\[ \text{where } ( \cdot ) \text{ is the binomial coefficient, } y = 0 \text{ or } 1, \text{ when } p = 2, \text{ and } y = 1, \text{ when } p > 2, \]

\( \text{for } p > 2, p n \geq m, \text{ and } y = 0 \text{ or } 1, \)

\[ \beta y p^m p^n = (1 - y) \sum_i (-1)^{m+i} \left( m - pi + (p - 1)(n - m + i - 1) \right) \beta P^{m+n-i} p^i \]

\[ - \sum_i (-1)^{m+i} \left( m - pi + (p - 1)(n - m + i - 1) \right) \beta y p^{m+n-i} \beta P p^i. \quad (2.11) \]

**Proof.** Consider the triple \( C = (E, A, F), \) where \( A \) is a cocommutative Hopf algebra over \( \mathbb{Z}/p, \) \( E \) and \( F \) are, respectively, the right and left cocommutative coalgebras over \( A. \) From the above discussion and considering the triple \( L = (A^e, k[\ell], k^e), \) then \( k[\ell], A^e, \) and \( k^e \) become, respectively, cocommutative Hopf algebra over \( \mathbb{Z}/p, \) and right and left cocommutative \( k[\ell]-\text{coalgebras, and} \)

\[ H^n(B(L)^*) = HC^n(A). \]

**Remark 2.4.** Note that if we replace the category \( k[\ell] \) by a reflexive category \( k[R] \) (see \([7, 9]\), then the Steenrod operations can be defined on the reflexive homology.
3. Some computations of Steenrod operations. We use operads and algebra of operads to obtain some computations of the Steenrod operations on the cohomology of a Hopf algebra over \( \mathbb{Z}/p \). Let \( H^* \) be the cohomology of the Hopf algebra \( A \) and consider the Steenrod operations

\[
P^i : H^{n}(S, H^*) \to H^{n+i}(S, H^*),
\]

where the algebra \( S \) over operad is the \( S_w \)-algebra structure over \( H^* \) and \( S_w = \{ S_w(j) \}_j \) is the cyclic operad generated by elements \( u_i \in S_w(2) \) and \( \pi_i \in S_w(i+2) \) (see [6]).

**Proposition 3.1.** There is an \( S_w \)-algebra over \( H^* \) generated by an element \( h_0 \) of dimension one such that \( \pi_i(h_0, h_1, \ldots, h_{i+1}) = 0 \), where \( h_i \) are given inductively by \( h_{i+1} = h_i P^1 h_i \).

**Lemma 3.2** [14]. Let \( X \) be a simplicial complex, \( CX \) the free commutative coalgebra generated by \( X \), \( A \) a Steenrod algebra where \( P^0 = 1 \), \( A[H^*(X)] \) a free unstable \( A \)-module generated by \( H^*(X) \), and \( S[A[H^*(X)]] \) a commutative algebra generated by \( A[H^*(X)] \) with multiplication given by \( x \cdot x = x \cup x \). Then \( H^*(CX) \cong S[A[H^*(X)]] \).

**Lemma 3.3.** There exists a chain equivalence \( B(SA, H^*) \cong B(A, (S, H^*)) \).

**Proof** (sketch). Let \( Y_* \) denote the cohomology of \( (SA, H^*) \). We then have the complex

\[
B(A, Y_*) : \cdots \to A^2 Y_* \to AY_* \to Y_*
\]

with the cohomology given by

\[
H^n(B(A, Y_*)) = \begin{cases} 
0 & n > 1, \\
H^i & n = 0.
\end{cases}
\]

The nontrivial cohomology group \( H^i \) is generated by elements \( \xi_i \), and \( Y_* \) is clearly a free unstable \( A \)-module with generator \( \xi_i \) and \( AY_* = H^n(B(S, H^*)) \) with generator \( \xi_{i+1} \).

**Proof of Proposition 3.1.** Consider the diagram

\[
\begin{array}{cccccc}
\cdots & \to & A \otimes A \otimes A & \to & A \otimes A & \to & A \\
& & \downarrow & \downarrow & \downarrow & \\
\cdots & \to & C(A \otimes A \otimes A) & \to & C(A \otimes A) & \to & C(A) \\
& & \downarrow & \downarrow & \downarrow & \\
\cdots & \to & C^2(A \otimes A \otimes A) & \to & C^2(A \otimes A) & \to & C^2(A) \\
& & \downarrow & \downarrow & \downarrow & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]
where $A$ is the Hopf algebra over $\mathbb{Z}/p$ and $C(A)$ is a free cocommutative coalgebra generated by $A$. The cohomology of the first row is by definition $H^*$. Consider the $B$ construction $B(S, H^*)$, where the differential is defined as $Sw$-algebra structure on $H^*$. The zero-dimensional cohomology of this $B$-construction contains the indecomposable elements in $H^*$ and also the elements

$$h_1 = h_0 p^1 h_0, \quad h_2 = h_1 p^1 h_1, \ldots, h_i = h_{i-1} p^1 h_{i-1},$$

$$h_i^2 = h_i p^0 h_i, \quad h_i^2 = h_i^2 p^0 h_i^2, \ldots, h_i^k = h_i^{k-1} p^0 h_i^{k-1}, \quad \text{where } h_i^k \in A^k. \quad (3.5)$$

Note that these elements are also indecomposable. The one-dimensional cohomology of $B(S, H^*)$ is a free unstable $A$-module with one generator $\xi_2^2 = h_0 h_1 \in S^1H^*$, which means that $h_0 h_1$ is acyclic ($\pi_i(h_0, h_1) = 0$). Consequently, the $i$-dimensional cohomology has one generator $\xi_{i+1} \in S^iH^*$, where $\xi_{i+1} = (h_0 \cdots h_{i+1})$. Hence $\pi_i(h_0 \cdots h_{i+1}) = 0$.

\textbf{Consequences.} From the above discussion, we conclude that the indecomposable elements in $H^*$ are $h^2 \in A^2$ and multiplication between these elements is given by the Cartan formula

$$(XY)P^n(XY) = \sum_{i=0}^{i=n} (XP^iX)(YP^{n-i}Y). \quad (3.6)$$

(a) Using the operation $P^2$ with $h_0 h_1 = 0$, we obtain $h_i h_{i+1} = 0$.

(b) Taking the operation $P^1$ and $h_i h_{i+1} = 0$, we obtain $h_i h_{i+k+2} = 0$ for any nonnegative integer $k$.

(c) If we use the operation $P^3$, we get the relations $h_i h_{i+k+2} = 0$ for any nonnegative integer $k$.

\textbf{References}


Mohamed Elhamdadi: Department of Mathematics, University of South Florida, Tampa, FL 33620, USA
E-mail address: emohamed@math.usf.edu

Yasien Gh. Gouda: Department of Mathematics, Faculty of Science, South Valley University, Aswan, Egypt
E-mail address: yasien10@hotmail.com