TWO-POINT DISTORTION THEOREMS FOR CERTAIN FAMILIES OF ANALYTIC FUNCTIONS IN THE UNIT DISC

YAŞAR POLATOĞLU, METİN BOLCAL, and ARZU ŞEN

Received 11 August 2002

We give two-point distortion theorems for various subfamilies of analytic univalent functions. We also find the necessary and sufficient condition for these subclasses of analytic functions.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let \( \Omega \) be the family of functions \( \omega(z) \) regular in the unit disc \( D = \{ z \mid |z| < 1 \} \) and satisfying the conditions \( \omega(0) = 0, |\omega(z)| < 1 \) for \( z \in D \).

For arbitrary fixed numbers \( A, B, -1 \leq B < A \leq 1 \), denote by \( P(A,B) \) the family of functions

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots
\]

(1.1)

regular in \( D \), such that \( p(z) \in P(A,B) \) if and only if

\[
p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},
\]

(1.2)

for some functions \( \omega(z) \in \Omega \) and every \( z \in D \). This class was introduced by Janowski [6].

Moreover, let \( C(A,B,b) \) denote the family of functions

\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots
\]

(1.3)

regular in \( D \), such that \( f(z) \in C(A,B,b) \) if and only if

\[
1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)},
\]

(1.4)

where \( b \neq 0 \), \( b \) is a complex number, for some functions \( p(z) \in P(A,B) \) and all \( z \in D \).

Next we consider the following class of functions defined in \( D \).

Let \( S^*(A,B,b) \) denote the family of functions

\[
f(z) = z + b_1 z + b_2 z^2 + b_3 z^3 + \cdots
\]

(1.5)
regular in $D$, such that $f(z) \in S^*(A,B,b)$ if and only if
\[
1 + \frac{1}{b} \left( \frac{z f'(z)}{f(z)} - 1 \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)},
\]
(1.6)
where $b \neq 0$, $b$ is a complex number, for some functions $\omega(z) \in \Omega$ and all $z \in D$.

We obtain the following subclasses of $C(A,B,b)$ by giving specific values to $A$, $B$, and $b$. For $A = 1$, $B = -1$, and $b = 1 - \alpha$ ($0 \leq \alpha < 1$), $C(1, -1, 1 - \alpha)$ is the class of convex functions of order $\alpha$ introduced by Robertson [9].

For $A = 1$, $B = -1$, and $b = (1 - \alpha)e^{-i\lambda}\cos\lambda$, $|\lambda| < \pi/2$, $C(1, -1, (1 - \alpha)e^{-i\lambda}\cos\lambda)$ is the class of functions for which $zf''(z)$ is $\lambda$-spirallike; this class was introduced by Robertson [10].

If we write
\[
1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) = C(f'(z), f''(z), b),
\]
then we obtain the following classes:

1. the class $C(1, 0, b)$ defined by $|C(f'(z), f''(z), b) - 1| < 1$,
2. the class $C(\beta, 0, b)$ defined by $|C(f'(z), f''(z), b) - 1| < \beta$, $0 \leq \beta < 1$,
3. the class $C(\beta, -\beta, b)$ defined by
\[
\left| \frac{C(f'(z), f''(z), b) - 1}{C(f'(z), f''(z), b) + 1} \right| < 1,
\]
$$0 < \beta,$$
(1.7)
4. the class $C(1, (1 - 1/M), b)$ defined by $|C(f'(z), f''(z), b) - M| < M$, $M > 1$.

Similarly, the subclasses of $S^*(A,B,b)$ are obtained by giving specific values to $A$, $B$, and $b$. These subclasses are obtained in [1, 2, 7, 8, 11].

2. Preliminary lemmas. For the purpose of this paper, we give the following lemmas.

**Lemma 2.1.** The necessary and sufficient condition for $f(z) \in C(A,B,b)$ is
\[
f(z) = \begin{cases} 
\int_{\zeta}^{z} (1 + B\omega(\zeta))^{b(A-B)/b} d\zeta, & B \neq 0 \\
\int_{0}^{z} e^{bA\omega(\zeta)} d\zeta, & B = 0,
\end{cases}
\]
(2.1)
where $\omega(z) \in \Omega$. 
THEOREMS FOR CERTAIN FAMILIES ... 4185

PROOF. Let \( B \neq 0 \) and let \( f(z) \in C(A,B,b) \). From the definition of the class \( C(A,B,b) \), we can write

\[
1 + \frac{1}{B} z \frac{f''(z)}{f'(z)} = p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}.
\] (2.2)

Equality (2.2) can be written in the form

\[
\frac{f''(z)}{f'(z)} = \frac{b(A-B)}{A} \frac{\omega'(z)}{1 + B\omega(z)}.
\] (2.3)

by using Jack’s lemma [5]. Integrating both sides of equality (2.3), we obtain

\[
f(z) = \int_0^z (1 + B\omega(\zeta))^{b(A-B)/B} d\zeta.
\] (2.4)

Equality (2.4) shows that \( f(z) \in C(A,B,b) \).

Conversely, if we take differentiation from equality (2.3), we obtain

\[
f'(z) = (1 + Bw(z))^{b(A-B)/B}.
\] (2.5)

Differentiating both sides of equality (2.5), we obtain

\[
z f''(z) = b(A-B) z \omega'(z) + B\omega(z).
\] (2.6)

Using Jack’s lemma [5] and after the simple calculations from (2.6), we obtain

\[
1 + \frac{1}{B} z \frac{f''(z)}{f'(z)} = p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}.
\] (2.7)

This equality shows that \( f(z) \in C(A,B,b) \). Similarly, we obtain

\[
f(z) = \int_0^z e^{bA\omega(\zeta)} d\zeta \iff f(z) \in C(A,B,b), \quad B = 0.
\] (2.8)

\[\square\]

**Lemma 2.2.** Let \( f(z) \in C(A,B,b) \) \( \Rightarrow z f'(z) \in S^*(A,B,b) \).

**Proof.** Let

\[
g(z) = z f'(z).
\] (2.9)

Taking a logarithmic derivative of (2.9), and after simple calculations, we get

\[
1 + \frac{1}{B} z \frac{g'(z)}{g(z)} = 1 + \frac{1}{B} z \frac{f''(z)}{f'(z)}.
\] (2.10)

This shows that the lemma is true. \[\square\]

**Lemma 2.3.** The class \( C(A,B,b) \) is invariant under the rotation so that \( f(e^{i\alpha} z) \in C(A,B,b) \), \( |\alpha| \leq 1 \), whenever \( f(z) \in C(A,B,b) \).
Proof. Let \( g(z) = f(e^{i\alpha}z) \). After the simple calculations from this equality we get

\[
1 + \frac{1}{b} z \frac{g''(z)}{g'(z)} = 1 + \frac{1}{b} \left( e^{i\alpha}z \right) \frac{f''(e^{i\alpha}z)}{f'(e^{i\alpha}z)}, \quad |\xi| = |e^{i\alpha}z| < 1. \quad (2.11)
\]

This shows that the lemma is true. \( \square \)

We note that the class \( S^*(A, B, b) \) is invariant under the rotation so that \( f(e^{i\alpha}z) \in S^*(A, B, b) \), \( |\alpha| \leq 1 \), whenever \( f(z) \in S^*(A, B, b) \).

Lemma 2.4. Let \( f(z) \) be regular and analytic in \( D \) and normalized so that \( f(0) = 0 \) and \( f'(0) = 1 \). A necessary and sufficient condition for \( f(z) \in C(A, B, b) \) is that for each member \( g(z) \), \( g(z) = z + a_2 z^2 + \cdots \), of \( S^*(A, B, b) \), the equation

\[
g(z) = z \left( \frac{f(z) - f(\zeta)}{z - \zeta} \right)^2, \quad z, \zeta \in D, z \neq \zeta, \zeta = \eta z, |\eta| \leq 1, \quad (2.12)
\]

must be satisfied.

Proof. Let \( f(z) \in C(A, B, b) \), then this function is analytic, regular, and continuous in the unit disc \( D \) and by using Lemmas 2.2 and 2.3, equality (2.12) can be written in the form

\[
g(z) = z \left( f'(z) \right)^2. \quad (2.13)
\]

Taking the logarithmic derivative from equality (2.13) and after simple calculations, we get

\[
1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = 1 + \frac{1}{2b} \left( z g'(z) \frac{g(z)}{g(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}. \quad (2.14)
\]

If we consider equality (2.14), the definition of \( C(A, B, b) \), and the definition of \( S^*(A, B, b) \), we obtain that \( g(z) \in S^*(A, B, b) \).

Conversely, let \( g(z) \in S^*(A, B, b) \), then on simple calculations from equality (2.12), we get

\[
1 + \frac{1}{b} \left( z g'(z) - 1 \right) = 1 + \frac{1}{b} \left[ \frac{2zf''(z)}{f(z) - f(\zeta)} \frac{z + \zeta}{z - \zeta} \right] + 1 - \frac{1}{b^2}. \quad (2.15)
\]

If we write

\[
F(z, \zeta) = \frac{1}{b} \left[ \frac{2zf''(z)}{f(z) - f(\zeta)} \frac{z + \zeta}{z - \zeta} \right] + 1 - \frac{1}{b^2}, \quad (2.16)
\]

equality (2.15) can be written in the form

\[
F(z, \zeta) = 1 + \frac{1}{b} \left( z g'(z) - 1 \right). \quad (2.17)
\]
On the other hand,

\[
\lim_{\zeta \to z} F(z, \zeta) = 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = 1 + \frac{1}{b} \left( \frac{z g'(z)}{g(z)} - 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.18}
\]

Equality (2.18) shows that \( f(z) \in C(A,B,b) \).

**Corollary 2.5.** If \( f(z) \in C(A,B,b) \), then

\[
2 \left[ 1 + \frac{1}{b} \left( z \frac{f'(z)}{f(z)} - 1 \right) \right] - 1 = p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.19}
\]

**Proof.** If we take \( \zeta = 0 \) in \( F(z, \zeta) \), we obtain

\[
F(z,0) = \frac{1}{b} \left( 2 z \frac{f'(z)}{f(z)} - 1 \right) + 1 - \frac{1}{b} = p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.20}
\]

This shows that the corollary is true.

**Corollary 2.6.** If \( f(z) \in C(A,B,b) \), then the set of values of \( (z f'(z)/f(z)) \) is the closed disc with centre \( C(r) \) and radius \( g(r) \), where

\[
C(r) = \frac{2 - 2B^2 + |b| (AB - B^2)}{2(1 - B^2 r^2)}, \quad g(r) = \frac{|b| (A - B) r}{2(1 - B^2 r^2)}. \tag{2.21}
\]

The proof of this corollary is obtained by using (2.19) and the inequality

\[
\left| p(z) - \frac{1 - ABr^2}{1 - B^2 r^2} \right| \leq \frac{(A - B) r}{1 - B^2 r^2}, \quad p(z) \in P(A,B). \tag{2.22}
\]

Inequality (2.22) was proved by Janowski [6].

**Lemma 2.7.** If \( f(z) \in C(A,B,b) \) and \( h_\rho(z) \) is defined by

\[
h_\rho(z) = \frac{f((z + a)/(1 + \rho \overline{a})) - f(a)}{(1 - |a|^2) f'(a)}, \quad a, z \in D, \quad \rho \in (0,1), \tag{2.23}
\]

then \( h_\rho(z) \in C(A,B,b) \).
**Proof.** Let $B \neq 0$. After simple calculations from (2.23), we obtain

\[
1 + \frac{1}{b} z \frac{h''(z)}{H(z)} = \frac{(1 - |a|^2)z}{(z+a)(1+z\bar{a})} \left[ 1 + \frac{1}{b} \left( \rho \left( \frac{z+a}{1+z\bar{a}} \right) \right) \frac{f''(\rho((z+a)/(1+z\bar{a})))}{f'(\rho((z+a)/(1+z\bar{a})))} \right] \]

(2.24)

\[
+ \left[ 1 - \frac{1}{b} \frac{2z\bar{a}}{1+z\bar{a}} - \frac{(1 - |a|^2)z}{(z+a)(1+z\bar{a})} \right].
\]

On the other hand, if we use Lemma 2.1, we can write

\[
\frac{f(\rho((z+a)/(1+z\bar{a}))) - f(a)}{(1 - |a|^2)f'(a)} = \int_{0}^{z} (1 + B\omega(\zeta)) \frac{b(\rho - B)}{B} d\zeta.
\]

(2.25)

After a brief computation from equality (2.25), we get

\[
1 + A\omega(z) = \left( \frac{1 - |a|^2}{(z+a)(1+z\bar{a})} \right) \left[ 1 + \frac{1}{b} \left( \rho \left( \frac{z+a}{1+z\bar{a}} \right) \right) \frac{f''(\rho((z+a)/(1+z\bar{a})))}{f'(\rho((z+a)/(1+z\bar{a})))} \right]
\]

(2.26)

\[
+ \left[ 1 - \frac{1}{b} \frac{2z\bar{a}}{1+z\bar{a}} - \frac{(1 - |a|^2)z}{(z+a)(1+z\bar{a})} \right].
\]

Let $B = 0$. Similarly,

\[
\frac{f(\rho((z+a)/(1+z\bar{a}))) - f(a)}{(1 - |a|^2)f'(a)} = \int_{0}^{z} e^{h(x)} dx \Rightarrow 1 + A\omega(z) = 1 + A\omega(z)
\]

(2.27)

In (2.26) and (2.27), letting $z = e^{i\theta}$ and $\omega = \rho((e^{i\theta} + a)/(1 + e^{i\theta}\bar{a}))$ gives

\[
1 + A\omega(z) = \frac{1 - |a|^2}{1 + ae^{-i\theta}|^2} \left[ 1 + \frac{1}{b} \omega \frac{f''(\omega)}{f'(\omega)} \right]
\]

(2.28)

\[
+ \left[ 1 - \frac{1}{b} \frac{2e^{i\theta}\bar{a}}{1 + e^{i\theta}\bar{a}} - \frac{(1 - |a|^2)e^{i\theta}}{1 + e^{-i\theta}a \bar{a}} \right].
\]
and we conclude that \( h_\rho(z) \) is in (2.27) for every admissible \( \rho \). From the compactness of \( C(A, B, b) \) and (2.28), we infer that \( h(z) = \lim_{\rho \to 1} h_\rho(z) \) is in \( C(A, B, b) \).

3. Two-point distortion for the class \( C(A, B, b) \). In this section, we give two-point distortion theorems for the class \( C(A, B, b) \).

**Theorem 3.1.** Let \( f(z) \in C(A, B, b) \). Then for \( |z| = r, 0 \leq r < 1 \),

\[
\frac{(1 + B|z|)^{(B - A)(|b| - \text{Re}b)/2B}}{(1 - B|z|)^{(B - A)(|b| + \text{Re}b)/2B}} \leq |f''(z)| \leq \frac{(1 - B|z|)^{(B - A)(|b| - \text{Re}b)/2B}}{(1 + B|z|)^{(B - A)(|b| + \text{Re}b)/2B}}, \quad B \neq 0,
\]

\[e^{-A|b||z|} \leq |f'(z)| \leq e^{A|b||z|}, \quad B = 0.\]  

**Proof.** If we use the definition of the class \( C(A, B, b) \), then we obtain

\[
\text{Re} \left( z \frac{f'''(z)}{f'(z)} \right) \geq \frac{\text{Re} b(B^2 - AB)r^2 - |b|(A - B)r}{1 - B^2r^2}, \quad B \neq 0,
\]

since

\[
\text{Re} \left( z \frac{f''(z)}{f'(z)} \right) = r \frac{\partial}{\partial r} \log |f'(z)|, \quad |z| = r;
\]

and using (3.2), we obtain

\[
\frac{\partial}{\partial r} \log |f'(z)| \geq \frac{\text{Re} b(B^2 - AB)r - |b|(A - B)}{(1 - B^2r^2)}.\]

Integrating both sides of inequality (3.4) from 0 to \( r \), we obtain

\[
|f'(z)| \geq \frac{(1 + B|z|)^{(B - A)(|b| - \text{Re}b)/2B}}{(1 - B|z|)^{(B - A)(|b| + \text{Re}b)/2B}},
\]

Similarly, we obtain the bounds on the right-hand side of (3.1).

If \( B = 0 \), then we have

\[-|b|Ar \leq \text{Re} z \frac{f'''(z)}{f'(z)} \leq |b|Ar;\]
and using (3.3), we obtain

\[ -|b|A \leq \frac{\partial}{\partial r} \log |f'(z)| \leq |b|A. \tag{3.7} \]

Integrating both sides of inequality (3.7) from 0 to \( r \), we obtain the desired result.

**Theorem 3.2.** If \( f(z) \in C(A,B,b) \), then, for \( |z| = r, 0 \leq r < 1 \),

\[
\frac{|z|(1 + B|z|)}{(1 - B|z|)} \left( \frac{(B - A)((|b| - \text{Re} b)) / 4B}{(B - A)((|b| + \text{Re} b)) / 4B} \right) \leq |f(z)| \leq \frac{|z|e^{-|b|A/2}}{2} \leq |f(z)| \leq |z|e^{|b|A|z|/2}, \quad B \neq 0. \tag{3.8}
\]

**Proof.** If we use Corollaries 2.5 and 2.6 and the definition of the classes \( C(A,B,b) \) and \( P(A,B) \), we can write

\[
\left\| 2 \left( 1 + \frac{1}{b} \left( z^2 \frac{f''(z)}{f'(z)} - 1 \right) - 1 \right) \right\| \leq \frac{1 - AB r^2}{1 - B^2 r^2} \tag{3.9}
\]

After the simple calculations from inequality (3.9), we get

\[
\text{Re} z \frac{f''(z)}{f'(z)} \geq \frac{2 - |b|(A - B)r - (2B^2 - (B^2 - AB) \text{Re} b)r^2}{1 - B^2 r^2} \tag{3.10}
\]

since

\[
\text{Re} z \frac{f''(z)}{f'(z)} = r \frac{\partial}{\partial r} \log |f(z)|; \tag{3.11}
\]

and using (3.10), we obtain

\[
\frac{\partial}{\partial r} \log |f(z)| \geq \frac{2 - |b|(A - B)r - (2B^2 - (B^2 - AB) \text{Re} b)r^2}{2r(1 - B^2 r^2)}. \tag{3.12}
\]

Integrating both sides of this inequality from 0 to \( r \), we obtain

\[
|f(z)| \geq \frac{|z|(1 + B|z|)}{(1 - B|z|)} \left( \frac{(B - A)((|b| - \text{Re} b)) / 4B}{(B - A)((|b| + \text{Re} b)) / 4B} \right). \tag{3.13}
\]

Similarly, we obtain the upper bounds in (3.8). Thus we end the proof. \( \square \)
We note that the bounds in Theorems 3.1 and 3.2 are sharp because the extremal function is

\[ f_*(z) = \begin{cases} e^{Abz}, & B \neq 0, \\ z(1-Bz)^{(B-A)(|b|-2Reb)/4B} \frac{1+Bz}{(1-Bz)^{(B-A)(|b|+Reb)/4B}}, & B = 0, \end{cases} \]

\[ z = \left( \frac{r(r-\sqrt{b/b})}{1-r\sqrt{b/b}} \right). \]  

(3.14)

**Corollary 3.3.** Let \( f(z) \in C(A,B,b) \). Then

\[ \alpha F_1(u,v) \leq |f(u) - f(v)| \leq \alpha F_2(u,v), \quad B \neq 0, \]

\[ \alpha G_1(u,v) \leq |f(u) - f(v)| \leq \alpha G_2(u,v), \quad B = 0, \]  

(3.15)

where

\[ \alpha = (1 - |v|^2) \frac{|u - v|}{|1 - uv|}, \]

\[ F_1(u,v) = \frac{(1+B|z|)^{3(B-A)(|b|-Reb)/4B} (B-A)(|b|+Reb)/4B}}{(1-B|z|)^{3(B-A)(|b|-Reb)/4B}}, \]

\[ F_2(u,v) = \frac{(1-B|z|)^{3(B-A)(|b|-Reb)/4B}}{(1+B|z|)^{3(B-A)(|b|+Reb)/4B}}, \]

\[ G_1(u,v) = e^{-(3/2)|b|A|u-v|/|1-uv|}, \]

\[ G_2(u,v) = e^{(3/2)|b|A|u-v|/|1-uv|}. \]  

(3.16)

**Proof.** If we consider Lemmas 2.1 and 2.7 and Theorem 3.2, then we can write

\[ |z|(1+B|z|)\frac{(B-A)(|b|-Reb)/4B}{(1-B|z|)^{(B-A)(|b|+Reb)/4B}} \leq \left| \frac{f((z+a)/(1+z\bar{a})) - f(a)}{(1-|a|^2)f'(a)} \right| \]

\[ \leq |z|(1-B|z|)\frac{(B-A)(|b|-Reb)/4B}{(1+B|z|)^{(B-A)(|b|+Reb)/4B}}, \quad B \neq 0, \]  

(3.17)

\[ |z|e^{-|b|A|z|^2/2} \leq \left| \frac{f((z+a)/(1+z\bar{a})) - f(a)}{(1-|a|^2)f'(a)} \right| \]

\[ \leq |z|e^{-|b|A|z|^2/2}, \quad B = 0. \]
Inequalities (3.17) can be written in the form
\[
(1 - |a|^2) |f'(a)| M_1(|z|) \leq \left| f\left( \frac{z + a}{1 + z\overline{a}} \right) - f(a) \right| \\
\leq (1 - |a|^2) |f'(a)| M_2(|z|), \quad B \neq 0,
\]
(3.18)
\[
(1 - |a|^2) |f'(a)| N_1(|z|) \leq \left| f\left( \frac{z + a}{1 + z\overline{a}} \right) - f(a) \right| \\
\leq (1 - |a|^2) |f'(a)| N_2(|z|), \quad B = 0,
\]
where
\[
M_1(|z|) = \frac{|z|(1 + B|z|)}{(1 - B|z|)} \left( B - A \right) \left( |b| - \frac{2|Re b|}{4B} \right), \\
M_2(|z|) = \frac{|z|(1 - B|z|)}{(1 + B|z|)} \left( B - A \right) \left( |b| + \frac{2|Re b|}{4B} \right),
\]
(3.19)
\[
N_1(|z|) = |z| e^{-|b|A|z|/2}, \\
N_2(|z|) = |z| e^{-|b|A|z|/2}.
\]

If we take \( v = a, u = (z + v)/(1 + z\overline{v}) \), or \( z = (u - v)/(1 - u \cdot \overline{v}) \), and if we use Theorem 3.1 in inequalities (3.18), we obtain the desired result.

We note that these inequalities are sharp because the extremal function is
\[
f_*(z) = \begin{cases} 
  e^{Abz}, & B \neq 0 \\
  z(1 - Bz)^\left( B - A \right) \left( |b| - \frac{2|Re b|}{4B} \right), & B = 0.
\end{cases}
\]
(3.20)

REFERENCES


Yaşar Polatoglu: Department of Mathematics and Computer Science, Faculty of Sciences and Arts, Istanbul Kültür University, Istanbul 34191, Turkey
E-mail address: y.polatoglu@iku.edu.tr

Metin Bolcal: Department of Mathematics and Computer Science, Faculty of Sciences and Arts, Istanbul Kültür University, Istanbul 34191, Turkey
E-mail address: m.bolcal@iku.edu.tr

Arzu Şen: Department of Mathematics and Computer Science, Faculty of Sciences and Arts, Istanbul Kültür University, Istanbul 34191, Turkey