ON A CLASS OF HOLOMORPHIC FUNCTIONS DEFINED BY THE RUSCHEWEYH DERIVATIVE

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By using the Ruscheweyh operator \( D_m f(z) \), \( z \in U \), we will introduce a class of holomorphic functions, denoted by \( M_m^\alpha(\alpha) \), and obtain some inclusion relations.

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1. Introduction and preliminaries. Denote by \( U \) the unit disc of the complex plane

\[
U = \{ z \in \mathbb{C}; |z| < 1 \}. \tag{1.1}
\]

Let \( \mathcal{H}(U) \) be the space of holomorphic functions in \( U \).
We let

\[
A_n = \{ f \in \mathcal{H}(U), f(z) = z + a_{n+1} z^{n+1} + \cdots, z_1 \in U \} \tag{1.2}
\]

with \( A_1 = A \).
We let \( \mathcal{H}[a, n] \) denote the class of analytic functions in \( U \) of the form

\[
f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad z \in U. \tag{1.3}
\]

If \( f \) and \( g \) are analytic in \( U \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \) or \( f(z) \prec g(z) \), if there is a function \( w \) analytic in \( U \), with \( w(0) = 0, |w(z)| < 1 \), for any \( z \in U \), such that \( f(z) = g(w(z)) \), for \( z \in U \).

If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subset g(U) \).
Let \( K = \{ f \in A : \text{Re}(zf''(z)/f'(z)) + 1 > 0, z \in U \} \) denote the class of normalized convex functions in \( U \). We use the following subordination results.

**Lemma 1.1** (Miller and Mocanu [2, page 71]). Let \( h \) be a convex function with \( h(0) = a \) and let \( \gamma \in \mathbb{C}^* \) be a complex with \( \text{Re} \gamma \geq 0 \). If \( p \in \mathcal{H}[a, n] \) and

\[
p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \tag{1.4}
\]
then \( p(z) < g(z) < h(z) \), where

\[
g(z) = \frac{Y}{n z^{y/n}} \int_0^z h(t) \cdot t^{(y/n)-1} dt.
\] (1.5)

The function \( g \) is convex and is the best \((a,n)\) dominant.

**Lemma 1.2** (Miller and Mocanu [1]). Let \( g \) be a convex function in \( U \) and let

\[
h(z) = g(z) + n \alpha zg'(z),
\] (1.6)

where \( \alpha > 0 \) and \( n \) is a positive integer. If \( p(z) = g(0) + p_n z^n + \cdots \) is holomorphic in \( U \) and

\[
p(z) + \alpha z p'(z) \prec h(z),
\] (1.7)

then

\[
p(z) \prec g(z)
\] (1.8)

and this result is sharp.

**Definition 1.3** [4]. For \( f \in A \) and \( m \geq 0 \), the operator \( D^m f \) is defined by

\[
D^m f(z) = f(z) \ast \frac{z}{(1-z)^{m+1}} = \frac{Z}{m!} \left[ z^{m-1} f(z) \right]^{(m)}, \quad z \in U,
\] (1.9)

where \( \ast \) stands for convolution.

**Remark 1.4.** We have

\[
D^0 f(z) = f(z), \quad z \in U,
\]

\[
D^1 f(z) = z f'(z), \quad z \in U,
\]

\[
2D^2 f(z) = z \cdot [D^1 f(z)]' + D^1 f(z),
\]

\[
(m + 1)D^{m+1} f(z) = z[D^m f(z)]' + m D^m f(z).
\] (1.10)

2. Main results

**Definition 2.1.** If \( \alpha < 1 \) and \( m, n \in \mathbb{N} \), let \( M^m_n(\alpha) \) denote the class of functions \( f \in A_n \) which satisfy the inequality

\[
\text{Re} \left( D^m f \right)'(z) > \alpha.
\] (2.1)

**Theorem 2.2.** If \( \alpha < 1 \) and \( m, n \in \mathbb{N} \), then

\[
M^m_{n+1}(\alpha) \subset M^m_n(\delta),
\] (2.2)
ON A CLASS OF HOLOMORPHIC FUNCTIONS

where

\[ \delta = \delta(\alpha, n, m) = 2\alpha - 1 + 2 \cdot (1 - \alpha) \cdot \frac{m + 1}{n} \beta\left(\frac{m + 1}{n}\right), \]

\[ \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt. \] (2.3)

PROOF. Let \( f \in M_{n}^{m+1}(\alpha) \). By using the properties of the operator \( D^m f(z) \), we have

\[ (m + 1)D^{m+1}f(z) = z \cdot (D^m f)'(z) + mD^m f(z), \quad z \in U. \] (2.4)

Differentiating (2.4), we obtain

\[ (m + 1)[D^{m+1}f(z)]' = z \cdot (D^m f)''(z) + (D^m f)'(z) + m(D^m f)'(z) \]

\[ = z(D^m f)''(z) + (m + 1)(D^m f)'(z). \] (2.5)

If we let \( p(z) = (D^m f)'(z) \), then \( p'(z) = (D^m f)''(z) \) and (2.4) becomes

\[ [D^{m+1}f(z)]' = p(z) + \frac{1}{m+1}z \cdot p'(z). \] (2.6)

Since \( f \in M_{n}^{m+1}(\alpha) \), by using Definition 2.1, we have

\[ \text{Re}\left[ p(z) + \frac{1}{m+1}z p'(z) \right] > \alpha \] (2.7)

which is equivalent to

\[ p(z) + \frac{1}{m+1}z p'(z) < \frac{1 + (2\alpha - 1)z}{1+z} \equiv h(z). \] (2.8)

By using Lemma 1.1, we have

\[ p(z) < g(z) < h(z), \] (2.9)

where

\[ g(z) = \frac{m + 1}{nz^{(m+1)/n}} \int_0^z \frac{1 + (2\alpha - 1)t}{1+t} \cdot t^{(m+1)/n-1} dt. \] (2.10)

The function \( g \) is convex and is the best dominant.

From \( p(z) < g(z) \), it results that

\[ \text{Re} p(z) > \delta = g(1) = \delta(\alpha, n, m), \] (2.11)
where
\[
g(1) = \frac{m+1}{n} \int_0^1 t^{(m+1)/n} \cdot \frac{\frac{1}{1+t} + (2\alpha - 1)t}{1+t} \, dt
\]
\[
= 2\alpha - 1 + 2 \cdot \frac{m+1}{n} \cdot (1-\alpha) \beta \left( \frac{m+1}{n} \right),
\]
(2.12)
from which we deduce that \( M_{n}^{m+1}(\alpha) \subset M_{n}^{m}(\delta) \).

For \( n = 1 \), this result was obtained in [3].

**Theorem 2.3.** Let \( g \) be a convex function, \( g(0) = 1 \), and let \( h \) be a function such that

\[
h(z) = g(z) + \frac{1}{m+1} z g'(z).
\]
(2.13)

If \( f \in A_n \) and verifies the differential subordination

\[
(D^{m+1} f)'(z) < h(z),
\]
(2.14)
then

\[
(D^{m} f)'(z) < g(z).
\]
(2.15)

**Proof.** From

\[
(m+1)D^{m+1} f(z) = z \cdot (D^{m} f)'(z) + mD^{m} f(z),
\]
(2.16)
we obtain

\[
(m+1)\left[ D^{m+1} f(z) \right]' = (D^{m} f)'(z) + z(D^{m} f)''(z) + m(D^{m} f)'(z)
\]
\[
= z(D^{m} f)''(z) + (m+1)(D^{m} f)'(z).
\]
(2.17)
If we let \( p(z) = (D^{m} f)'(z) \), then we obtain

\[
[D^{m+1} f(z)]' = p(z) + \frac{1}{m+1} z p'(z)
\]
(2.18)
and (2.14) becomes

\[
p(z) + \frac{1}{m+1} z p'(z) < g(z) + \frac{1}{m+1} z g'(z) \equiv h(z).
\]
(2.19)

By using Lemma 1.2, we have

\[
p(z) < g(z), \quad \text{i.e., } (D^{m} f)'(z) < g(z).
\]
(2.20)

For \( n = 1 \), this result was obtained in [3].
ON A CLASS OF HOLOMORPHIC FUNCTIONS ... 4143

**Theorem 2.4.** Let \( h \in \mathcal{H}[U] \), with \( h(0) = 1 \), \( h'(0) \neq 0 \), which verifies the inequality

\[
\Re \left[ 1 + \frac{zh''(z)}{h'(z)} \right] > -\frac{1}{2(m+1)}, \quad m \geq 0.
\]

(2.21)

If \( f \in A_n \) and verifies the differential subordination

\[
[D^{m+1}f(z)]' < h(z), \quad z \in U,
\]

(2.22)

then

\[
[D^mf(z)]' < g(z),
\]

(2.23)

where

\[
g(z) = \frac{m+1}{nz^{(m+1)/n}} \int_0^z h(t)t^{(m+1)/n-1} dt.
\]

(2.24)

The function \( g \) is convex and is the best dominant.

**Proof.** A simple application of the differential subordination technique [1, 2] shows that the function \( g \) is convex. From

\[
(m+1)D^{m+1}f(z) = z[D^mf(z)]' + mD^mf(z),
\]

(2.25)

we obtain

\[
(m+1)[D^{m+1}f(z)]' = z[D^mf(z)]'' + (m+1)[D^mf(z)]'.
\]

(2.26)

If we let \( p(z) = [D^mf(z)]' \), then we obtain

\[
[D^{m+1}f(z)]' = p(z) + \frac{1}{m+1}zp'(z)
\]

(2.27)

and (2.22) becomes

\[
p(z) + \frac{1}{m+1}zp'(z) < h(z).
\]

(2.28)

By using Lemma 1.1, we have

\[
p(z) < g(z) = \frac{m+1}{nz^{(m+1)/n}} \int_0^z h(t)t^{(m+1)/n-1} dt.
\]

(2.29)

**Theorem 2.5.** Let \( g \) be a convex function, \( g(0) = 1 \), and

\[
h(z) = g(z) + nzg'(z).
\]

(2.30)

If \( f \in A_n \) and verifies the differential subordination

\[
[D^mf(z)]' < h(z), \quad z \in U,
\]

(2.31)
then

$$\frac{D^m f(z)}{z} < g(z). \quad (2.32)$$

**Proof.** We let \( p(z) = D^m f(z)/z, z \in U, \) and we obtain

$$D^m f(z) = zp(z). \quad (2.33)$$

By differentiating, we obtain

$$[D^m f(z)]' = p(z) + zp'(z), \quad z \in U. \quad (2.34)$$

Then (2.31) becomes

$$p(z) + zp'(z) < h(z) = g(z) + zg'(z). \quad (2.35)$$

By using Lemma 1.2, we have (1.8). \( \square \)

**References**


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