Let $H$ be a finite-dimensional Hopf algebra over a field $k$, $H^*$ the dual Hopf algebra of $H$, and $B$ a right $H^*$-Galois and Hirata separable extension of $B^H$. Then $B$ is characterized in terms of the commutator subring $V_B(B^H)$ of $B^H$ in $B$ and the smash product $V_B(B^H)#H$. A sufficient condition is also given for $B$ to be an $H^*$-Galois Azumaya extension of $B^H$.

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1. Introduction. Let $H$ be a finite-dimensional Hopf algebra over a field $k$, $H^*$ the dual Hopf algebra of $H$, and $B$ a right $H^*$-Galois extension of $B^H$. In [3], the class of $H^*$-Galois Azumaya extensions was investigated and in [8], it was shown that $B$ is a Hirata separable extension of $B^H$ if and only if the commutator subring $V_B(B^H)$ of $B^H$ in $B$ is a left $H$-Galois extension of $C$, where $C$ is the center of $B$ (see [8, Lemma 2.1, Theorem 2.6]). The purpose of the present paper is to characterize a right $H^*$-Galois and Hirata separable extension $B$ of $B^H$ in terms of the commutator subring $V_B(B^H)$ and the smash product $V_B(B^H)#H$. Let $B$ be a right $H^*$-Galois extension of $B^H$ such that $B^H = B^{H^*}$. Then the following statements are equivalent:

1. $B$ is a Hirata separable extension of $B^H$,
2. $V_B(B^H)$ is an Azumaya $C$-algebra and $V_B(V_B(B^H)) = B^H$,
3. $V_B(B^H)$ is a right $H^*$-Galois extension of $C$ and a direct summand of $V_B(B^H)#H$ as a $V_B(B^H)$-bimodule,
4. $V_B(B^H)$ is a right $H^*$-Galois extension of $C$ and $V_B(B^H)#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

Moreover, an equivalent condition is given for a right $H^*$-Galois and Hirata separable extension $B$ of $B^H$ to be an $H^*$-Galois Azumaya extension which was studied in [3, 7]. Also, let $B$ be a right $H^*$-Galois and Hirata separable extension of $B^H$ and $A$ a subalgebra of $B^H$ over $C$ such that $B^H$ is a projective Hirata separable extension of $A$ containing $A$ as a direct summand as an $A$-bimodule. Then $V_{B^H}(A)$ is a separable subalgebra of $B^H$ over $C$, and there exists an $H$-submodule algebra $D$ in $B$ which is separable over $C$ such that $D^H = V_{B^H}(A)$ and $D \equiv V_{B^H}(A) \otimes_Z F$ as Azumaya $Z$-algebras, where $Z$ is the center of $D$ and $F$ is an Azumaya $Z$-algebra in $D$. 
2. Basic definitions and notations. Throughout, $H$ denotes a finite-dimensional Hopf algebra over a field $k$ with comultiplication $\Delta$ and counit $\varepsilon$, $H^*$ the dual Hopf algebra of $H$. $B$ is a left $H$-module algebra, $C$ the center of $B$. $B^H = \{ b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H \}$ which is called the $H$-invariants of $B$, and $B \# H$ the smash product of $B$ with $H$, where $B \# H = B \otimes_k H$ such that for all $b \# h$ and $b' \# h'$ in $B \# H$, $(b \# h)(b' \# h') = \sum b(h_1 b')\# h_2 h'$, where $\Delta(h) = \sum h_1 \otimes h_2$. The ring $B$ is called a right $H^*$-Galois extension of $B^H$ if $B$ is a right $H^*$-comodule algebra with structure map $\rho : B \to B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \to B \otimes_k H^*$ is a bijection, where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring $A$ of $B$ with the same identity 1, we denote the commutator subring of $A$ in $B$ by $V_B(A)$. We call $B$ a separable extension of $A$ if there exist $\{a_i, b_i \}$ in $B$, $i = 1, 2, \ldots, m$, for some integer $m$ such that $\sum a_i b_i = 1$ and $\sum b_i a_i = \sum a_i b_i b$ for all $b$ in $B$, where $\otimes$ is over $A$. An Azumaya algebra is a separable extension of its center. $B$ is a left Hopf $A$-module. $A$ is called an $H^*$-Galois extension, if $B$ is separable over $B^H$ which is an $A$-Azumaya algebra over $C^H$. A right $H^*$-Galois extension $B$ of $B^H$ is called an $H^*$-Galois Hirata extension if $B$ is also a Hirata separable extension of $B^H$. Throughout, an $H^*$-Galois extension of $B^H$ means a right $H^*$-Galois extension unless it is stated otherwise.

3. The $H^*$-Galois Hirata extensions. In this section, we will characterize an $H^*$-Galois Hirata extension $B$ of $B^H$ in terms of the commutator subring $V_B(B^H)$ of $B^H$ in $B$ and the smash product $V_B(B^H) \# H$. A relationship between an $H^*$-Galois Hirata extension and an $H^*$-Galois Azumaya extension is also given. We begin with some properties of an $H^*$-Galois Hirata extension $B$ of $B^H$. Throughout, we assume $B^H = B^{H^*}$.

**Lemma 3.1.** If $A_1$ and $A_2$ are $H^*$-Galois extensions such that $A_1^H = A_2^H$ and $A_1 \subset A_2$, then $A_1 = A_2$.

**Proof.** By [3, Theorem 5.1], there exist $\{x_i, y_i \in A_1 \mid i = 1, 2, \ldots, n\}$ for some integer $n$ such that, for all $h \in H$, $\sum x_i(h y_i) = T(h)1_{A_1}$, where $T \in \mathbb{I}_{H^*}$, the set of right integrals in $H^*$. Let $t \in \mathbb{I}_H$ be left integrals in $H$, such that $T(t) = 1$, then $\{x_i, f_i = t(y_i) \mid i = 1, 2, \ldots, n\}$ is a dual basis of the finitely generated and projective right module $A_1$ over $A_1^H$. Since $A_1 \subset A_2$ such that $A_1^H = A_2^H$, $\{x_i, f_i \mid i = 1, 2, \ldots, n\}$ is also a dual basis of the finitely generated and projective right module $A_2$ over $A_2^H$. This implies that $A_1 = A_2$. \qed

**Lemma 3.2.** If $B$ is an $H^*$-Galois Hirata extension of $B^H$, then $B^H$ is a direct summand of $B$ as a $B^H$-bimodule.

**Proof.** We use the argument as given in [2]. Since $B$ is an $H^*$-Galois and a Hirata separable extension of $B^H$, $V_B(B^H)$ is a left $H$-Galois extension of $C$ (see [8, Lemma 2.1, Theorem 2.6]). Hence, $V_B(B^H)$ is a finitely generated and
projective module over \( C \) (see [3, Theorem 2.2]). Let \( \Omega = \text{Hom}_C(\mathcal{V}_B(B^H), \mathcal{V}_B(B^H)) \). Since \( C \) is commutative, \( \mathcal{V}_B(B^H) \) is a pre-generator of \( C \). Thus, \( B \) is a right \( \Omega \)-module such that \( B \cong \mathcal{V}_B(B^H) \otimes_C \text{Hom}_\Omega(\mathcal{V}_B(B^H), B) \cong \mathcal{V}_B(B^H) \otimes_C B^{H*} \) as \( C \)-algebras, where \( f(1) \in B^{H*} \) for each \( f \in \text{Hom}_\Omega(\mathcal{V}_B(B^H), B) \) by the proof of [2, Lemma 2.8]. But \( \mathcal{V}_B(B^H) = B^H \) (see [2, Lemma 2.5]), so \( B \cong \mathcal{V}_B(B^H) \otimes_C B^H \). This implies that \( \mathcal{V}_B(B^H) \) is an \( h^*-\)Galois extension of \( C \) (see [2, Lemma 2.8]); and so \( C \) is a direct summand of \( \mathcal{V}_B(B^H) \) as a \( C \)-bimodule (see [2, Corollaries 1.9 and 1.10]). Therefore, \( B^H \) is a direct summand of \( B \) as a \( B^H \)-bimodule.

By the proof of Lemma 3.2, \( \mathcal{V}_B(B^H) \) is an \( h^*-\)Galois extension of \( C \).

**Corollary 3.3.** If \( B \) is an \( h^*-\)Galois Hirata extension of \( B^H \), then \( \mathcal{V}_B(B^H) \) is an \( h^*-\)Galois extension of \( C \).

**Corollary 3.4.** If \( B \) is an \( h^*-\)Galois Hirata extension of \( B^H \), then \( B = B^H \cdot \mathcal{V}_B(B^H) \) and the centers of \( B \), \( B^H \), and \( \mathcal{V}_B(B^H) \) are the same \( C \).

**Proof.** By Corollary 3.3, \( \mathcal{V}_B(B^H) \) is an \( h^*-\)Galois extension of \( C \), so \( B^H \cdot \mathcal{V}_B(B^H) \) is also an \( h^*-\)Galois extension of \( B^H \) (\( = (B^H \cdot \mathcal{V}_B(B^H))^H \)) with the same Galois system as \( \mathcal{V}_B(B^H) \) (see [3, Theorem 5.1]). Noting that \( B^H \cdot \mathcal{V}_B(B^H) \subset B \), we conclude that \( B = B^H \cdot \mathcal{V}_B(B^H) \) by Lemma 3.1. Moreover, \( \mathcal{V}_B(\mathcal{V}_B(B^H)) = B^H \) (see [8, Lemma 2.5]), so the centers of \( B^H \), \( \mathcal{V}_B(B^H) \), and \( B \) are the same \( C \).}

**Theorem 3.5.** Let \( B \) be an \( h^*-\)Galois extension of \( B^H \). The following statements are equivalent:

1. \( B \) is a Hirata separable extension of \( B^H \),
2. \( \mathcal{V}_B(B^H) \) is an \( h^*-\)Galois extension of \( C \) and a direct summand of \( \mathcal{V}_B(B^H) \# H \) as a \( \mathcal{V}_B(B^H) \)-bimodule,
3. \( \mathcal{V}_B(B^H) \) is an Azumaya \( C \)-algebra and \( \mathcal{V}_B(\mathcal{V}_B(B^H)) = B^H \),
4. \( \mathcal{V}_B(B^H) \) is an \( h^*-\)Galois extension of \( C \) and \( \mathcal{V}_B(B^H) \# H \) is a direct summand of a finite direct sum of \( \mathcal{V}_B(B^H) \) as a bimodule over \( \mathcal{V}_B(B^H) \).

**Proof.** (1)\(\Rightarrow\)(3). Since \( B \) is an \( h^*-\)Galois and a Hirata separable extension of \( B^H \), by Lemma 3.2, \( B^H \) is a direct summand of \( B \) as a \( B^H \)-bimodule. Thus, \( \mathcal{V}_B(\mathcal{V}_B(B^H)) = B^H \) and \( \mathcal{V}_B(B^H) \) is a separable \( C \)-algebra (see [4, Propositions 1.3 and 1.4]). But the center of \( \mathcal{V}_B(B^H) \) is \( C \) by Corollary 3.4, so \( \mathcal{V}_B(B^H) \) is an Azumaya \( C \)-algebra.

(3)\(\Rightarrow\)(1). Since \( \mathcal{V}_B(B^H) \) is an Azumaya \( C \)-algebra and \( B \) is a bimodule over \( \mathcal{V}_B(B^H) \), \( B \cong \mathcal{V}_B(B^H) \otimes_C \mathcal{V}_B(\mathcal{V}_B(\mathcal{V}_B(B^H))) = \mathcal{V}_B(B^H) \otimes_C B^H \) as a bimodule over \( \mathcal{V}_B(B^H) \) (see [1, Corollary 3.6, page 54]). Noting that \( B \cong \mathcal{V}_B(B^H) \otimes_C B^H \) is also an isomorphism as \( C \)-algebras and that \( \mathcal{V}_B(B^H) \) is an Azumaya \( C \)-algebra, we conclude that \( \mathcal{V}_B(B^H) \otimes_C B^H \) is a Hirata separable extension of \( B^H \); and so \( B \) is a Hirata separable extension of \( B^H \).

(3)\(\Rightarrow\)(2). By the proof of (3)\(\Rightarrow\)(1), \( B \cong \mathcal{V}_B(B^H) \otimes_C B^H \) such that \( \mathcal{V}_B(B^H) \) is a finitely generated and projective module over \( C \), so \( \mathcal{V}_B(B^H) \) is an \( h^*-\)Galois extension of \( C \) (see [2, Lemma 2.8]). Moreover, since \( \mathcal{V}_B(B^H) \) is an Azumaya
C-algebra, $V_B(B^H)$ is a direct summand of $V_B(B^H) \otimes_C (V_B(B^H))^\ast$ as a $V_B(B^H)$-bimodule, where $(V_B(B^H))^\ast$ is the opposite algebra of $V_B(B^H)$. But $V_B(B^H) \otimes_C (V_B(B^H))^\ast \cong \text{Hom}_C(V_B(B^H), V_B(B^H)) \cong V_B(B^H)\#H$ (see [3, Theorem 2.2]), so $V_B(B^H)$ is a direct summand of $V_B(B^H)\#H$ as a $V_B(B^H)$-bimodule.

(2)$\Rightarrow$(3). Since $V_B(B^H)$ is an $H^\ast$-Galois extension of $C$, $B^H \cdot V_B(B^H)$ is an $H^\ast$-Galois extension of $(B^H \cdot V_B(B^H))\#H$. But $(B^H \cdot V_B(B^H))\#H = B^H$, so $B^H \cdot V_B(B^H)$ and $B$ are $H^\ast$-Galois extensions of $B^H$ such that $B^H \cdot V_B(B^H) \subset B$. Hence, $B^H \cdot V_B(B^H) = B$ by Lemma 3.1. Thus, the centers of $B$ and $V_B(B^H)$ are the same $C$. Moreover, $V_B(B^H)$ is a direct summand of $V_B(B^H)\#H$ as a $V_B(B^H)$-bimodule by hypothesis, so it is a separable $C$-algebra (see [3, Theorem 2.3]). Thus, $V_B(B^H)$ is an Azumaya $C$-algebra. But then $B \cong V_B(B^H) \otimes_C V_B(B^H)$). On the other hand, by hypothesis, $V_B(B^H)$ is an $H^\ast$-Galois extension of $C$, so $B \cong V_B(B^H) \otimes_C V_B(B^H)$ (see [2, Lemma 2.8]). Therefore, $V_B(V_B(B^H)) = B^H$.

(3)$\Leftrightarrow$(4). Since $V_B(B^H)$ is an $H^\ast$-Galois extension of $C$, it is a finitely generated and projective module over $C$ and $\text{Hom}_C(V_B(B^H), V_B(B^H)) \cong V_B(B^H)\#H$ (see [3, Theorem 2.2]). But then $V_B(B^H)$ is a Hirata separable extension of $C$ if and only if $V_B(B^H)\#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$ (see [5, Corollary 3]). Thus, $V_B(B^H)$ is an Azumaya $C$-algebra if and only if $V_B(B^H)$ is an $H^\ast$-Galois extension of $C$ and $V_B(B^H)\#H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

By Theorem 3.5, we can obtain a relationship between the class of $H^\ast$-Galois Hirata extensions and the class of $H^\ast$-Galois Azumaya extensions which were studied in [3, 7].

**Corollary 3.6.** Let $B$ be an $H^\ast$-Galois Azumaya extension of $B^H$. Then $B$ is an $H^\ast$-Galois Hirata extension of $B^H$ if and only if $C = C^H$.

**Proof.** ($\Rightarrow$) Since $B$ is an $H^\ast$-Galois Hirata extension of $B^H$, $V_B(B^H)$ is an Azumaya algebra over $C$ and a left $H$-Galois extension of $C$ (see [8, Theorem 2.6]). Hence, $V_B(V_B(B^H)) = B^H$ (see [8, Lemma 2.5]). Thus, $C \subset B^H$; and so $C = C^H$.

($\Leftarrow$) Since $B$ is an $H^\ast$-Galois Azumaya extension of $B^H$, $V_B(B^H)$ is separable over $C^H$ (see [3, Lemma 4.1]). Since $B$ is an $H^\ast$-Galois Azumaya extension of $B^H$ again, $V_B(B^H)$ is an $H^\ast$-Galois extension of $(V_B(B^H))\#H$ (see [3, Lemma 4.1]), so both $B^H \cdot V_B(B^H)$ and $B$ are $H^\ast$-Galois extensions of $B^H$ such that $B^H \cdot V_B(B^H) \subset B$. Hence, $B^H \cdot V_B(B^H) = B$ by Lemma 3.1. This implies that the center of $V_B(B^H)$ is $C$. But by hypothesis, $C = C^H$, so $V_B(B^H)$ is an Azumaya $C$-algebra. Hence, $V_B(B^H)$ is a Hirata separable extension of $C$. But $B = B^H \cdot V_B(B^H) \cong B^H \otimes_C V_B(B^H)$ as Azumaya $C$-algebras, so $B$ is a Hirata separable extension of $B^H$. Thus, $B$ is an $H^\ast$-Galois Hirata extension of $B^H$.

**Corollary 3.7.** Let $B$ be an $H^\ast$-Galois Hirata extension of $B^H$. Then $B$ is an $H^\ast$-Galois Azumaya extension of $B^H$ if and only if $B$ is an Azumaya $C^H$-algebra.
Proof. \((\Rightarrow)\) Since \(B\) is an \(H^*-\text{Galois Azumaya extension of } BH\), \(BH\) is an Azumaya \(C^H\)-algebra and \(B\) is separable over \(BH\) (see [3, Theorem 3.4]). Hence, \(B\) is separable over \(C^H\) by the transitivity of separable extensions. But \(B\) is an \(H^*-\text{Galois Azumaya extension of } BH\) and an \(H^*-\text{Galois Hirata extension of } BH\) by hypothesis, so \(C = C^H\) by Corollary 3.6. This implies that \(B\) is an Azumaya \(C^H\)-algebra.

\((\Leftarrow)\) By hypothesis, \(B\) is an Azumaya \(C^H\)-algebra. Hence, \(C = C^H\). But \(B\) is an \(H^*-\text{Galois Hirata extension of } BH\) again, \(B\) is a Hirata separable extension of \(BH\) and a finitely generated and projective module over \(BH\). Thus, \(V_B(V_B(BH)) = BH\) (see [8, Lemma 2.5]); and so \(BH = V_B(V_B(BH))\) is an Azumaya subalgebra of \(B\) over \(C^H\) by the commutator theorem for Azumaya algebras (see [1, Theorem 4.3, page 57]). This proves that \(B\) is an \(H^*-\text{Galois Azumaya extension of } BH\).

4. Invariant subalgebras. For an \(H^*-\text{Galois Hirata extension } B\) as given in Theorem 3.5, let \(A\) be a subalgebra of \(BH\) over \(C\) such that \(BH\) is a projective Hirata separable extension of \(A\) and contains \(A\) as a direct summand as an \(A\)-bimodule. In this section, we show that \(V_B(A)\) is an \(H\)-invariant subalgebra of a separable subalgebra \(D\) in \(B\) over \(C\), that is, \(D^H = V_B(A)\). We denote by \(\mathcal{F}\) the set \(\{A \mid A\text{ is an }\mathcal{H}\text{-subalgebra of } BH\text{ over } C\}\) such that \(V_B(A)\) is a separable \(C\)-algebra (see Corollary 3.4 and [6, Theorem 1]).

Lemma 4.1. Let \(B\) be an \(H^*-\text{Galois Hirata extension of } BH\). For any \(A \in \mathcal{F}\), \(V_B(A)\) is an \(H\)-submodule algebra of \(B\) and separable over \(C\), and \((V_B(A))^H = V_B(A)\) which is a separable \(C\)-algebra.

Proof. Since \(A \in \mathcal{F}\), \(BH\) is a projective Hirata separable extension of \(A\) and contains \(A\) as a direct summand as an \(A\)-bimodule. But \(B\) is an \(H^*-\text{Galois Hirata extension of } BH\), so \(B\) is a projective Hirata separable extension of \(BH\). Hence, by the transitivity property of projective Hirata separable extensions, \(B\) is a projective Hirata separable extension of \(A\). Also \(BH\) is a direct summand of \(B\) as a \(BH\)-bimodule by Lemma 3.2, so \(A\) is a direct summand of \(B\) as an \(A\)-bimodule. Thus, \(V_B(A)\) is a separable algebra over \(C\) (see [6, Theorem 1]). Moreover, it is clear that \((V_B(A))^H = V_B(A)\), so \(V_B(A)\) is a separable \(C\)-algebra (see Corollary 3.4 and [6, Theorem 1]).

Next we want to show which separable subalgebra of \(BH\) over \(C\) is an \(H\)-invariant subring of an \(H\)-submodule algebra in \(B\). Let \(\mathcal{T} = \{E \subset B \mid E\text{ is a separable }C\text{-subalgebra of } BH\text{ and satisfies the double centralizer property in } BH\text{ such that } V_B(E) \in \mathcal{F}\}\). Next we show that for any \(E \in \mathcal{T}\), \(E\) is the \(H\)-invariant subring of an \(H\)-submodule algebra \(D\) in \(B\) which is separable over \(C\).

Theorem 4.2. Let \(E\) be in \(\mathcal{T}\). Then there exists an \(H\)-submodule algebra \(D\) in \(B\) which is separable over \(C\) such that \(D^H = E\).
Proof. Since $E$ is in $\mathcal{F}$, $V_B H(E)$ is in $\mathcal{F}$ such that $V_B H(V_B H(E)) = E$. Now by Lemma 4.1, $V_B H(V_B H(E))$ is an $H$-submodule algebra of $B$ and separable over $C$ such that $(V_B H(V_B H(E)))^H = V_B H(V_B H(E))$. But $V_B H(V_B H(E)) = E$, so

$$(V_B H(V_B H(E)))^H = E.$$ (4.1)

Let $D = V_B H(V_B H(E))$. Then $D$ satisfies the theorem.

By Theorem 4.2, we obtain an expression for the separable $H$-submodule algebra $D$ for a given $E$ in $\mathcal{F}$.

Corollary 4.3. By keeping the notations as given in Theorem 4.2, let $Z$ be the center of $E$. Then $D \cong E \otimes_Z V_D(E)$ as Azumaya $Z$-algebras.

Proof. Since $E$ satisfies the double centralizer property in $B^H$, $V_B H(V_B H(E)) = E$. Hence, the centers of $E$ and $V_B H(E)$ are the same $Z$. Similarly as given in the proof of Lemma 4.1, since $V_B H(E)$ is in $\mathcal{F}$, $B (= B^H \cdot V_B H(B^H))$ is a projective Hirata separable extension of $V_B H(E)$ and contains $V_B H(E)$ as a direct summand as a $V_B H(E)$-bimodule by the transitivity property of projective Hirata separable extensions and the direct summand conditions. Thus, $V_B H(E)$ satisfies the double centralizer property in $B$, that is, $V_B (V_B (V_B H(E))) = V_B H(E)$. This implies that the centers of $V_B H(E)$ and $V_B (V_B H(E))$ are the same. Therefore, $D$ and $E$ have the same center $Z$. Noting that $D$ and $E$ are separable $C$-algebras by Theorem 4.2, we conclude that $E (= D^H)$ is an Azumaya subalgebra of $D$ over $Z$; and so $D \cong E \otimes_Z V_D(E)$ as Azumaya $Z$-algebras (see [1, Theorem 4.3, page 57]).

Remark 4.4. When $B$ is an $H^*$-Galois Azumaya extension of $B^H$, the correspondence $A \rightarrow V_B (A)$ as given in Lemma 4.1 recovers the one-to-one correspondence between the set of separable subalgebras of $B^H$ and the set of $H^*$-Galois extensions in $B$ containing $V_B (B^H)$ as given in [3].

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