A COEFFICIENT INEQUALITY FOR THE CLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Received 31 August 2002

The aim of this paper is to give a coefficient inequality for the class of analytic functions in the unit disc $D = \{ z \mid |z| < 1 \}$.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let $\Omega$ be the family of functions $\omega(z)$ regular in the disc $D$ and satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in D$.

Next, for arbitrary fixed numbers $A$ and $B$, $-1 < A \leq 1$, $-1 \leq B < A$, denote by $P(A,B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (1.1)$$

regular in $D$ such that $p(z)$ is in $P(A,B)$ if and only if

$$p(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)} \quad (1.2)$$

for some function $\omega(z) \in \Omega$ and every $z \in D$. The class $P(A,B)$ was introduced by Janowski [3].

Moreover, let $S^*(A,B,b)$ ($b \neq 0$, complex) denote the family of functions

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots \quad (1.3)$$

regular in $D$ and such that $f(z)$ is in $S^*(A,B,b)$ if and only if

$$1 + \frac{1}{b} \left( z \frac{f'(z)}{f(z)} - 1 \right) = p(z) \quad (1.4)$$

for some $p(z)$ in $P(A,B)$ and all $z$ in $D$.

For the aim of this paper we need Jack's lemma [2]. “Let $\omega(z)$ be a regular in the unit disc with $\omega(0) = 0$, then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_1$, we can write $z_1 \omega'(z_1) = k \omega(z_1)$, where $k$ is real and $k \geq 1$.”
2. Coefficient inequality. The main purpose of this paper is to give sharp upper bound of the modulus of the coefficient $a_n$. Therefore, we need the following lemma.

**Lemma 2.1.** *The necessary and sufficient condition for $g(z) = z + a_2z^2 + \cdots$ belongs to $S^*(A,B,b)$ is*

$$g(z) \in S^*(A,B,b) \iff g(z) = \begin{cases} z \cdot (1 + B\omega(z))^{b(A-B)/B}, & B \neq 0, \\ z \cdot e^{bA\omega(z)}, & B = 0, \end{cases} \quad (2.1)$$

where $\omega(z) \in \Omega$.

**Proof.** The proof of this lemma is in four steps.

**Step 1.** Let $B \neq 0$ and

$$g(z) = z \cdot (1 + B\omega(z))^{b(A-B)/B}. \quad (2.2)$$

If we take the logarithmic derivative from equality (2.2), we obtain

$$\frac{1}{b} \left( z \cdot \frac{g'(z)}{g(z)} - 1 \right) = (A-B) \frac{z \cdot \omega'(z)}{1 + B\omega(z)}. \quad (2.3)$$

If we use Jack’s lemma [2] in equality (2.3), we get

$$\frac{1}{b} \left( z \cdot \frac{g'(z)}{g(z)} - 1 \right) = \frac{(A-B)\omega(z)}{1 + B\omega(z)}. \quad (2.4)$$

After the simple calculations from (2.4), we see that

$$1 + \frac{1}{b} \left( z \cdot \frac{g'(z)}{g(z)} - 1 \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)}. \quad (2.5)$$

Equality (2.5) shows that $g(z) \in S^*(A,B,b)$.

**Step 2.** Let $B = 0$ and

$$g(z) = z \cdot e^{bA\omega(z)}. \quad (2.6)$$

Similarly, we obtain

$$1 + \frac{1}{b} \left( z \frac{g'(z)}{g(z)} - 1 \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)} = 1 + A\omega(z). \quad (2.7)$$

This shows that $g(z) \in S^*(A,B,b)$. 
Step 3. Let \( g(z) \in S^*(A,B,b) \) and \( B \neq 0 \), then we have
\[
1 + \frac{1}{B} \left( z \frac{g'(z)}{g(z)} - 1 \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)}.
\] (2.8)
Equality (2.8) can be written in the form
\[
\frac{g'(z)}{g(z)} = \frac{b(A-B)(\omega(z)/z)}{1 + B\omega(z)} + \frac{1}{z}.
\] (2.9)
If we use Jack’s lemma (2.9), we obtain
\[
\frac{g'(z)}{g(z)} = \frac{b(A-B)\omega'(z)}{1 + B\omega(z)} + \frac{1}{z}.
\] (2.10)
Integrating both sides of equality (2.10), we get
\[
g(z) = z \cdot (1 + B\omega(z))^{b(A-B)/B}.
\] (2.11)
Step 4. Let \( g(z) \in S^*(A,B,b) \) and \( B = 0 \). Similarly, we obtain
\[
g(z) = z \cdot e^{bA\omega(z)}
\] (2.12)
which ends the proof. \( \square \)

We note that we choose the branch of \((1 + B\omega(z))^{b(A-B)/B}\) such that
\[
(1 + B\omega(0))^{b(A-B)/B} = 1 \quad \text{at } z = 0.
\] (2.13)

Theorem 2.2. If \( f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots \) belongs to \( S^*(A,B,b) \), then
\[
|a_n| \leq \prod_{k=0}^{n-2} \left| \frac{b(A-B) + kB}{k+1} \right| \text{ if } B \neq 0,
\] (2.14)
\[
|a_n| \leq \prod_{k=0}^{n-2} \left| \frac{bA}{k+1} \right| \text{ if } B = 0.
\] (2.14)

These bounds are sharp because the extremal function is
\[
f^*_+(z) = \begin{cases} 
\frac{z}{(1-B\delta z)^{-b(A-B)/B}}, & |\delta| = 1, \text{ if } B \neq 0, \\
ze^{bAz}, & \text{if } B = 0.
\end{cases}
\] (2.15)

Proof. Let \( B \neq 0 \). If we use the definition of the class \( S^*(A,B,b) \), then we write
\[
1 + \frac{1}{B} \left( z \frac{f'(z)}{f(z)} - 1 \right) = p(z).
\] (2.16)
Equality (2.16) can be written by using the Taylor expansion of \( f(z) \) and \( p(z) \) in the form
\[
z + 2a_2z^2 + 3a_3z^3 + \cdots + nan^nz + \cdots
= (z + a_2z^2 + \cdots + a_nz^n + \cdots)(1 + b p_1z + b p_2z^2 + \cdots + b p_nz^n + \cdots).
\]
(2.17)

Evaluating the coefficient of \( z^n \) in both sides of (2.17), we get
\[
na_n = a_n + bp_1a_{n-1} + bp_2a_{n-2} + \cdots + bp_{n-1}.
\]
(2.18)

on the other hand,
\[
|p_n| \leq (A - B).
\]
(2.19)

Inequality (2.19) was proved by Aouf [1]. If we consider the relations (2.18) and (2.19) together, then we obtain
\[
(n - 1)|a_n| \leq |b||A - B|(1 + |a_2| + |a_3| + \cdots + |a_{n-1}|),
\]
(2.20)

which can be written in the form
\[
|a_n| \leq \frac{1}{(n - 1)} \sum_{k=1}^{n-1} |b||A - B||a_k|, \quad |a_1| = 1.
\]
(2.21)

To prove (2.14), we will use the induction principle. Now, we consider inequalities (2.21) and
\[
|a_n| \leq \prod_{k=0}^{n-2} \frac{|b(A - B) + kB|}{k + 1}.
\]
(2.22)

The right-hand sides of these inequalities are the same because
(i) for \( n = 2 \),
\[
|a_n| \leq \frac{|b||A - B|}{(n - 1)} \sum_{k=1}^{n-1} |a_k|, \quad |a_1| = 1 \Rightarrow |a_2| \leq |b||A - B|,
\]
(2.23)
(ii) for $n = 3$,

$$|a_3| \leq \frac{|b||A - B|}{(n - 1)} \sum_{k=1}^{n-1} |a_k| = \frac{1}{2} |b||A - B|(1 + |a_2|) \quad \Rightarrow |a_3| \leq \frac{1}{2} |b|^2|A - B|^2 + \frac{1}{2} |b||A - B|,$$

$$|a_3| \leq \prod_{k=0}^{n-2} \frac{|b(A - B) + kB|}{k + 1} = |b||A - B| \frac{|b(A - B) + B|}{2} \quad (2.24)$$

$$\Rightarrow |a_3| \leq \frac{1}{2} |b||A - B|(|b||A - B| + |B|) \leq \frac{1}{2} |b||A - B|(|b||A - B| + 1) \quad \Rightarrow |a_3| \leq \frac{1}{2} |b|^2|A - B|^2 + \frac{1}{2} |b||A - B|.$$ 

Suppose that this result is true for $n = p$, then we have

$$|a_n| \leq \frac{|b||A - B|}{(n - 1)} \sum_{k=1}^{n-1} |a_k|, \quad (2.25)$$

$$|a_1| = 1 \Rightarrow |a_p| \leq \frac{|b||A - B|}{(p - 1)} (1 + |a_2| + |a_3| + \cdots + |a_{p-1}|),$$

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|b(A - B) + kB|}{k + 1} \quad \Rightarrow |a_p| \leq \prod_{k=0}^{p-2} \frac{|b(A - B) + kB|}{k + 1} \quad (2.26)$$

$$\Rightarrow |a_p| \leq \frac{1}{(p - 1)!} |b||A - B|(|b||A - B| + 1) (|b||A - B| + 2) \cdot (|b||A - B| + 3) \cdots (|b||A - B| + (p - 2))$$

from (2.25), (2.26), and induction hypothesis, we have

$$\frac{|b||A - B|}{(p - 1)} (1 + |a_2| + |a_3| + \cdots + |a_{p-1}|) = \frac{1}{(p - 1)!} |b||A - B|(|b||A - B| + 1) \quad (2.27)$$

$$\cdot (|b||A - B| + 2) \cdots (|b||A - B| + (p - 2)).$$

If we write $x = |b||A - B| > 0$, equality (2.27) can be written in the form.

$$\frac{x}{(p - 1)} (1 + |a_2| + |a_3| + \cdots + |a_{p-1}|) = \frac{1}{(p - 1)!} x(x + 1)(x + 2) \cdots (x + (p - 2)). \quad (2.28)$$
After the simple calculation from equality (2.28), we get
\[
\frac{1}{p} (x + (p-1)) \left( 1 + |a_2| + |a_3| + \cdots + |a_{p-1}| \right) = \frac{1}{p!} (x+1)(x+2)(x+3) \cdots (x+(p-2))(x+(p-1))
\]
\[
\Rightarrow \frac{1}{p} \left[ \frac{x}{p-1} \right] (1 + |a_2| + |a_3| + \cdots + |a_{p-1}|) \right]\]
\[
+ \left[ \frac{1}{p} \left( 1 + |a_2| + |a_3| + \cdots + |a_{p-1}| \right) \right]
\]
\[
= \frac{1}{p!} (x+1)(x+2)(x+3) \cdots (x+(p-2))(x+(p-1))
\]
\[
\Rightarrow \frac{1}{p} |a_p| + \left[ \frac{1}{p} (1 + |a_2| + |a_3| + \cdots + |a_{p-1}|) \right]
\]
\[
= \frac{1}{p!} (x+1)(x+2)(x+3) \cdots (x+(p-2))(x+(p-1))
\]
\[
\Rightarrow \frac{x}{p} (1 + |a_2| + |a_3| + \cdots + |a_{p-1}| + |a_p|)
\]
\[
= \frac{1}{p!} x(x+1)(x+2)(x+3) \cdots (x+(p-2))(x+(p-1)).
\]

Equality (2.29) shows that the result is valid for \( n = p + 1 \).

Therefore, we have (2.14).

\[\square\]

**Corollary 2.3.** The first inequality of (2.14) can be rewritten in the form
\[
|a_n| \leq \prod_{k=0}^{n-2} \frac{|b(A-B) + kb|}{k+1}
\]
\[
= |B(A-B)| \left( \frac{1}{2} \right) |b(A-B) + B| \right.
\]
\[
\cdot \frac{1}{3} |b(A-B) + 2B| \cdots \frac{1}{(n-1)} |b(A-B) + (n-2)B| \right.
\]
\[
= \frac{1}{(n-1)!} |b(A-B)| \cdot |b(A-B) + B| \right.
\]
\[
\cdot |b(A-B) + 2B| \cdots |b(A-B) + (n-2)B| \right.
\]
\[
\leq \frac{1}{(n-1)!} |b(A-B)| \cdot (|b(A-B)| + |B|) \right.
\]
\[
\cdot (|b(A-B)| + 2|B|) \cdots (|b(A-B)| + (n-2)|B|). \quad (2.30)
\]

If \( A = 1, B = -1, \) and \( b = 1, \) then
\[
|a_n| \leq \frac{1}{(n-1)!} 2 \cdot (2+1) \cdot (2+2) \cdots n = \frac{n!}{(n-1)!} = n. \quad (2.31)
\]

This is the coefficient inequality for the starlike function which is well known.
Corollary 2.4. If $A = 1$, $B = -1$,

$$|a_n| < \frac{1}{(n-1)!} \prod_{k=0}^{n-2} |2b+k|.$$  \hspace{1cm} (2.32)

This inequality was obtained by Aouf [1].

Therefore, by giving the special value to $A$, $B$, and $b$, we obtain the coefficient inequality for the classes $S^*(1, -1, \beta)$, $S^*(1, -1, e^{-i\lambda}\cos\lambda)$, $S^*(1, -1, (1 - \beta)e^{-i\lambda}\cos\lambda)$, $S^*(1, 0, b)$, $S^*(\beta, 0, b)$, $S^*(\beta, -\beta, b)$, $S^*(1, (-1 + 1/M), b)$, and $S^*(1 - 2\beta, -1, b)$, where $0 \leq \beta < 1$, $|\lambda| < \pi/2$, and $M > 1$.

References


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