SOME VERSIONS OF ANDERSON’S AND MAHER’S INEQUALITIES I

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We prove the orthogonality (in the sense of Birkhoff) of the range and the kernel of an important class of elementary operators with respect to the Schatten \( p \)-class.

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1. Introduction. Let \( H \) be a separable infinite-dimensional complex Hilbert space and let \( B(H) \) denote the algebra of all bounded operators on \( H \) into itself. Given \( A,B \in B(H) \), we define the generalized derivation \( \delta_{A,B} : B(H) \to B(H) \) by

\[
\delta_{A,B}(X) = AX - XB
\]

and the elementary operator derivation \( \Delta_{A,B} : B(H) \to B(H) \) by

\[
\Delta_{A,B}(X) = AXB - X.
\]

Denote \( \delta_{A,A} = \delta_A \) and \( \Delta_{A,A} = \Delta_A \).

In [1, Theorem 1.7], Anderson shows that if \( A \) is normal and commutes with \( T \), then, for all \( X \in B(H) \),

\[
\| T + \delta_A(X) \| \geq \| T \|. \quad (1.1)
\]

It is shown in [10] that if the pair \( (A,B) \) has the Fuglede-Putnam property (in particular, if \( A \) and \( B \) are normal operators) and \( AT = TB \), then, for all \( X \in B(H) \),

\[
\| T + \delta_{A,B}(X) \| \geq \| T \|. \quad (1.2)
\]

Duggal [4] showed that the above inequality (1.2) is also true when \( \delta_{A,B} \) is replaced by \( \Delta_{A,B} \). The related inequality (1.1) was obtained by the author [11] showing that if the pair \( (A,B) \) has the Fuglede-Putnam property \((FP)_{C_p}\), then

\[
\| T + \delta_{A,B}(X) \|_p \geq \| T \|_p \quad (1.3)
\]

for all \( X \in B(H) \), where \( C_p \) is the von Neumann-Schatten class, \( 1 \leq p < \infty \), and \( \| \cdot \|_p \) is its norm for all \( X \in B(H) \) and for all \( T \in C_p \cap \ker \delta_{A,B} \). In all of the above results, \( A \) was not arbitrary. In fact, certain normality-like assumptions have been imposed on \( A \). A characterization of \( T \in C_p \) for \( 1 < p < \infty \), which is orthogonal to \( R(\delta_A|C_p) \) (the range of \( \delta_A|C_p \)) for a general operator \( A \), has
been carried out by Kittaneh [7], showing that if $T$ has the polar decomposition $T = U|T|$, then

$$
||T + \delta_A(X)||_p \geq ||T||_p
$$

for all $X \in C_p \ (1 < p < \infty)$ if and only if $|T|^{p-1}U^* \in \ker \delta_A$. By a simple modification in the proof of the above inequality, we can prove that this inequality is also true in the general case, that is, if $T$ has the polar decomposition $T = U|T|$, then $||T + \delta_{A,B}(X)||_p \geq ||T||_p$ for all $X \in C_p \ (1 < p < \infty)$ if and only if $|T|^{p-1}U^* \in \ker \delta_{B,A}$. In Sections 1, 2, 3, and 4, we prove these results in the case where we consider $E_{A,B}$ instead of $\delta_{A,B}$, which leads us to prove that if $T \in C_p$ and $\ker E_{A,B} \subseteq \ker E^*_{A,B}$, then

$$
||T + E_{A,B}(X)||_p \geq ||T||_p
$$

for all $X \in C_p \ (1 < p < \infty)$ if and only if $T \in \ker E_{A,B}$. In Sections 5 and 6, we minimize the map $||S + E_{A,B}(X)||_p$ and we classify its critical points.

2. Preliminaries. Let $T \in B(H)$ be compact and let $s_1(X) \geq s_2(X) \geq \cdots \geq 0$ denote the singular values of $T$, that is, the eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in their decreasing order. The operator $T$ is said to belong to the Schatten $p$-class $C_p$ if

$$
||T||_p = \left[ \sum_{i=1}^{\infty} s_i(T)^p \right]^{1/p} = \left[ \text{tr}(T)^p \right]^{1/p}, \quad 1 \leq p < \infty,
$$

where tr denotes the trace functional. Hence, $C_1$ is the trace class, $C_2$ is the Hilbert-Schmidt class, and $C_\infty$ is the class of compact operators with

$$
||T||_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|
$$

denoting the usual operator norm. For the general theory of the Schatten $p$-classes, the reader is referred to [8, 13].

Recall that the norm $\|\cdot\|$ of the $B$-space $V$ is said to be Gateaux differentiable at nonzero elements $x \in V$ if

$$
\lim_{t \to 0, t \in \mathbb{R}} \frac{\|x + ty\| - \|x\|}{t} = \mathbb{R}D_x(y)
$$

for all $y \in V$. Here $\mathbb{R}$ denotes the set of reals, $\mathbb{R}$ denotes the real part, and $D_x$ is the unique support functional (in the dual space $V^*$) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gateaux differentiability of the norm at $x$ implies that $x$ is a smooth point of the sphere of radius $\|x\|$.
It is well known (see [8] and the references therein) that, for \( 1 < p < \infty \), \( C_p \) is a uniformly convex Banach space. Therefore, every nonzero \( T \in C_p \) is a smooth point and, in this case, the support functional of \( T \) is given by

\[
D_T(X) = \text{tr} \left\{ \frac{|T|^{p-1}UX^*}{\|T\|_p^{p-1}} \right\} \tag{2.4}
\]

for all \( X \in C_p \), where \( T = U|T| \) is the polar decomposition of \( T \).

**Definition 2.1.** Let \( E \) be a complex Banach space. We define the orthogonality in \( E \) as follows: \( b \in E \) is orthogonal to \( a \in E \) if, for all complex \( \lambda \), there holds

\[
\|a + \lambda b\| \geq \|a\|. \tag{2.5}
\]

This definition has a natural geometric interpretation, namely, \( b \perp a \) if and only if the complex line \( \{a + \lambda b \mid \lambda \in \mathbb{C}\} \) is disjoint with the open ball \( K(0, \|a\|) \), that is, if and only if this complex line is a tangent one. Note that if \( b \) is orthogonal to \( a \), then \( a \) needs not be orthogonal to \( b \). If \( E \) is a Hilbert space, then from (2.5), it follows that \( \langle a, b \rangle = 0 \), that is, orthogonality in the usual sense.

3. **Main results.** In this section, we characterize \( T \in C_p \) for \( 1 < p < \infty \), which is orthogonal to \( R(\Delta_{A,B} | C_p) \) (the range of \( \Delta_{A,B} | C_p \)) for a general pair of operators \( A, B \).

**Lemma 3.1** [7]. Let \( u \) and \( v \) be two elements of a Banach space \( V \) with norm \( \| \cdot \| \). If \( u \) is a smooth point, then \( D_u(v) = 0 \) if and only if

\[
\|u + zv\| \geq \|u\| \tag{3.1}
\]

for all \( z \in \mathbb{C} \) (the complex numbers).

**Theorem 3.2.** Let \( A, B \in B(H) \) and \( T \in C_p \) (\( 1 < p < \infty \)). Then

\[
\|T + \Delta_{A,B}(X)\|_p \geq \|T\|_p \tag{3.2}
\]

for all \( X \in B(H) \) with \( \Delta_{A,B}(X) \in C_p \) if and only if \( \text{tr}(|T|^{p-1}U^*\Delta_{A,B}(X)) = 0 \) for all such \( X \).

**Proof.** The theorem is an immediate consequence of equality (2.4) and Lemma 3.1.

**Theorem 3.3.** Let \( A, B \in B(H) \) and \( T \in C_p \) (\( 1 < p < \infty \)). Then

\[
\|T + \Delta_{A,B}(X)\|_p \geq \|T\|_p \tag{3.3}
\]

for all \( X \in C_p \) if and only if \( \tilde{T} = |T|^{p-1}U^* \in \ker \Delta_{B,A} \).
**Proof.** By virtue of Theorem 3.2, it is sufficient to show that \( \text{tr}(T \Delta_{A,B}(X)) = 0 \) for all \( X \in C_p \) if and only if \( \tilde{T} \in \ker \Delta_{B,A} \).

Choose \( X \) to be the rank-one operator \( f \otimes g \) for some arbitrary elements \( f \) and \( g \) in \( H \); then \( \text{tr}(T(AXB - X)) = \text{tr}((B\tilde{T}A - T)X) = 0 \) implies that \( \langle \Delta_{B,A}(\tilde{T})f, g \rangle = 0 \) if and only if \( \tilde{T} \in \ker \Delta_{B,A} \). Conversely, assume that \( \tilde{T} \in \ker \Delta_{B,A} \), that is, \( B\tilde{T}A = \tilde{T} \).

Since \( \tilde{T}X \) and \( T\Delta_{B,A} \) are trace classes for all \( X \in C_p \), we get

\[
\text{tr}(\tilde{T}(AXB - X)) = \text{tr}(TAXB - \tilde{T}X) = \text{tr}(XB\tilde{T}A - XT) = 0.
\]

**Lemma 3.4.** Let \( A, B \in B(H) \) and \( S \in B(H) \) such that \( \ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*} \). If \( AU|S|^{p-1}B = U|S|^{p-1} \), where \( p > 1 \) and \( S = U|S| \) is the polar decomposition of \( S \), then \( AU|S|B = U|S| \).

**Proof.** If \( T = |S|^{p-1} \), then

\[
AUTB = UT. \tag{3.5}
\]

We prove that \( AUT^n B = UT^n \). \tag{3.6}

If \( ATB = T = A^*TB^* \), then \( BT^*T = BT^*ATB = T^*TB \), and thus \( B|T| = |T|B \) and \( BT^2 = T^2B \). Since \( B \) commutes with the positive operator \( T^2 \), then \( B \) commutes with its square roots, that is,

\[
BT = TB. \tag{3.7}
\]

By (3.5) and (3.7) we obtain (3.6). Let \( f(t) \) be the map defined on \( \sigma(T) \subset R^+ \) by

\[
f(t) = t^{1/(p-1)}, \quad 1 < p < \infty. \tag{3.8}
\]

Since \( f \) is the uniform limit of a sequence \( (P_i) \) of polynomials without constant term (since \( f(0) = 0 \)), it follows from (3.3) that \( AUP_i(T)B = UP_i(T) \). Therefore, \( AUT^{1/(p-1)}B = UT^{1/(p-1)} \). \( \square \)

**Theorem 3.5.** Let \( A \) and \( B \) be operators in \( B(H) \) such that \( \ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*} \). Then \( T \in \ker \Delta_{A,B} \cap C_p \) if and only if

\[
||S + \Delta_{A,B}(X)||_p \geq ||S||_p \tag{3.9}
\]

for all \( X \in C_p \).
Theorem 3.6. Let $A, B \in B(H)$. If

1. $A, B \in \mathcal{L}(H)$ such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in \mathcal{H}$,
2. $A$ is invertible and $B$ is such that $\|A^{-1}\| \|B\| \leq 1$,
3. $A = B$ is a cyclic subnormal operator,

then, $T \in \ker \Delta_{A,B} \cap C_p$ if and only if

$$\|S + \Delta_{A,B}(X)\|_p \geq \|S\|_p$$

for all $X \in C_p$.

Proof. The result of Tong [14, Lemma 1] guarantees that the above condition implies that for all $T \in \ker (\delta_{A,B}[\mathcal{H}(\mathcal{H})])$, $\overline{R(T)}$ reduces $A$, $\ker (T)^\perp$ reduces $B$, and $A|_{\overline{R(T)}}$ and $B|_{\ker (T)^\perp}$ are unitary operators. Take $\mathcal{H}_1 = \mathcal{H} = \overline{\text{ran } S} \oplus \overline{\text{ran } S}^\perp$ and $\mathcal{H}_2 = \mathcal{H} = \ker S \oplus \ker S^\perp$. According to the decomposition of $\mathcal{H}$ and for $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$, $A_2 : \mathcal{H}_2 \to \mathcal{H}_2$, and $S : \mathcal{H}_2 \to \mathcal{H}_1$, we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (3.13)

From $ASB = S$, it follows that $A_1SB_1 = S$, and since $A_1$ and $B_1$ are unitary operators, we obtain $A_1^*SB_1^* = S$, and the result holds by the above theorem.

The above inequality holds in particular if $A = B$ is isometric; in other words, $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$.

(2) In this case, it suffices to take $A_1 = \|B\|^{-1}A$ and $B_1 = \|B\|^{-1}B$, then $\|A_1x\| \geq \|x\| \geq \|B_1x\|$, and the result holds by (1) for all $x \in \mathcal{H}$.

(3) Since $T$ commutes with $A$, it follows that $T$ is subnormal [15]. But any compact subnormal operator is normal; hence, $T$ is normal. By applying Fuglede-Putnam theorem, we get that $ATA = T$ implies $A^*TA^* = T$. \hfill $\square$

4. The case where $n > 1$. Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be $n$-tuples of operators in $B(H)$. In this section, we characterize $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(E_{A,B}|C_p)$ (the range of $E_{A,B}|C_p$) for a general pair of operators $A$ and $B$. \hfill $\square$
By the same argument used in the proofs of Theorems 3.2 and 3.3, we prove the following theorems.

**Theorem 4.1.** Let \( A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) be \( n \)-tuples of operators in \( B(H) \) and \( T \in C_p \) \((1 < p < \infty)\). Then

\[
\left\| T + E_{A,B}(X) \right\|_p \geq \| T \|_p
\]

(4.1)

for all \( X \in B(H) \) with \( E_{A,B}(X) \in C_p \) if and only if \( \text{tr}(|T|^{p-1}U^*E_{A,B}(X)) = 0 \) for all such \( X \).

**Theorem 4.2.** Let \( A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) be \( n \)-tuples of operators in \( B(H) \) and \( T \in C_p \) \((1 < p < \infty)\). Then

\[
\left\| T + E_{A,B}(X) \right\|_p \geq \| T \|_p
\]

(4.2)

for all \( X \in C_p \) if and only if \( \tilde{T} = |T|^{p-1}U^* \in \ker E_{A,B} \).

**Lemma 4.3.** Let \( C = (C_1, C_2, \ldots, C_n) \) be \( n \)-tuple of operators in \( B(H) \) such that

\[
\sum_{i=1}^n C_i C_i^* \leq 1, \quad \sum_{i=1}^n C_i^* C_i \leq 1, \quad \text{and } \ker E_C \subseteq \ker E_C^*.
\]

If

\[
\sum_{i=1}^n C_i U|S|^{p-1} C_i = U|S|^{p-1},
\]

(4.3)

where \( p > 1 \) and \( S = U|S| \) is the polar decomposition of \( S \), then

\[
\sum_{i=1}^n C_i U|S| C_i = U|S|.
\]

(4.4)

**Proof.** If \( T = |S|^{p-1} \), then

\[
\sum_{i=1}^n C_i U T C_i = U T.
\]

(4.5)

We prove that

\[
\sum_{i=1}^n C_i U T^n C_i = U T^n.
\]

(4.6)

It is known that if \( \sum_{i=1}^n C_i C_i^* \leq 1, \sum_{i=1}^n C_i^* C_i \leq 1, \) and \( \ker E_C \subseteq \ker E_C^* \), then the eigenspaces corresponding to distinct nonzero eigenvalues of the compact positive operator \( |S|^2 \) reduce each \( C_i \) (see [3, Theorem 8], [14, Lemma 2.3]). In particular, \( |S| \) commutes with \( C_i \) for all \( 1 \leq i \leq n \). This implies also that \( |S|^{p-1} = T \) commutes with each \( C_i \) for all \( 1 \leq i \leq n \). Hence \( C_i |T| = |T| C_i \) and \( C_i T^2 = T^2 C_i \).
Since $C_i$ commutes with the positive operator $T^2$, then $C_i$ commutes with its square roots, that is,

$$C_i T = TC_i.$$  \hfill (4.7)

By the same arguments used in the proof of Lemma 3.4, the proof of this lemma can be completed.

**Theorem 4.4.** Let $C = (C_1, C_2, \ldots, C_n)$ be $n$-tuple of operators in $B(H)$ such that $\sum_{i=1}^{n} C_i C_i^* \leq \mathbf{1}$, $\sum_{i=1}^{n} C_i^* C_i \leq \mathbf{1}$, and $\ker E_C \subseteq \ker E_C^*$. Then $S \in \ker E_C \cap C_p$ $(1 < p < \infty)$ if and only if

$$\|S + E_C(X)\|_p \geq \|S\|_p$$  \hfill (4.8)

for all $X \in C_p$.

**Proof.** If $S \in \ker E_C$, then, from [14, Theorem 2.4], it follows that $\|S + E_C(X)\|_p \geq \|S\|_p$ for all $X \in C_p$. Conversely, if $\|S + E_C(X)\|_p \geq \|S\|_p$ for all $X \in C_p$, then, from Theorem 4.2, $\sum_{i=1}^{n} C_i |S|^{p-1} U^* C_i = |S|^{p-1} U^*$. Since $\ker E_C \subseteq \ker E_C^*$, $\sum_{i=1}^{n} C_i^* |S|^{p-1} U^* C_i^* = |S|^{p-1} U^*$. Taking adjoints, we get $\sum_{i=1}^{n} C_i U \times |S|^{p-1} C_i = U|S|^{p-1}$, and from Lemma 4.3, it follows that $\sum_{i=1}^{n} C_i U |S| C_i = U|S|$, that is, $S \in \ker E_C$.

**Theorem 4.5.** Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be $n$-tuples of operators in $B(H)$ such that $\sum_{i=1}^{n} A_i A_i^* \leq \mathbf{1}$, $\sum_{i=1}^{n} A_i^* A_i \leq \mathbf{1}$, $\sum_{i=1}^{n} B_i B_i^* \leq \mathbf{1}$, $\sum_{i=1}^{n} B_i^* B_i \leq \mathbf{1}$, and $\ker E_{A,B} \subseteq \ker E_{A^*,B^*}$.

Then $T \in \ker E_{A,B} \cap C_p$ if and only if

$$\|S + E_{A,B}(X)\|_p \geq \|S\|_p$$  \hfill (4.9)

for all $X \in C_p$.

**Proof.** It suffices to take the Hilbert space $H \oplus H$ and the operators

$$C_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \quad S = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$$  \hfill (4.10)

and apply Theorem 4.4.

**5. Remarks.** (1) It is known (see [8] and the references therein) that the smooth points of $K(H)$ are those compact operators that attain their norm at a unique (up to multiplication by a constant of modulus one) unit vector. It has been shown in [8] that a nonzero $T \in B(H)$ is a smooth point if and only if $T$ attains its norm at a unique (up to multiplication by a constant of modulus one) unit vector $e \in H$ and $\|T\|_e \leq \|T\|$, where $\|T\|_e$ is the essential
norm of $T$, that is, the norm of $\pi(T)$, where $\pi$ is the quotient map of $B(H)$ onto $B(H)/K(H)$. In this case,

$$D_T(X) = \text{tr} \left[ \frac{(e \otimes T e)}{\|T\|} X \right] = \left\langle X e, \frac{T e}{\|T\|} \right\rangle$$

(5.1)

for all $X \in B(H)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $H$ and $e \otimes T e$ is the rank-one operators defined by $(e \otimes T e)f = \langle f, Te \rangle e$ for all $f \in H$.

Hence, for the usual operator norm, Theorems 3.2, 3.3, 4.1, and 4.2 can be combined in the following formulation. Let $A,B \in B(H)$ and $T \in B(H)$ be a smooth point. If $T = e \otimes T e$, then the following statements are equivalent:

(i) $\|T + E_{A,B}(X)\| \geq \|T\|$ for all $X \in B(H)$,

(ii) $\text{tr}(T - E_{A,B}(X)) = 0$ for all $X \in B(H)$,

(iii) $T \in \ker E_{A,B}$.

(2) It is still possible to give a characterization similar to this given in the usual operator norm for the norm $\|\cdot\|_\infty$. However, in this case, we have to assume that $T$ is a smooth point, that is, the given norm is Gateaux differentiable at $T$ and $\tilde{T} = e \otimes T e$, where $e$ is the unique (up to multiplication by a constant of modulus one) unit vector at which $T$ attains its norm.

(3) It is well known that the Hilbert-Schmidt class $C_2$ is a Hilbert space under the inner product $\langle Y, Z \rangle = \text{tr} Z^* Y$.

We remark here that, for the Hilbert Schmidt norm $\|\cdot\|_2$, the orthogonality results in Theorems 3.3, 3.5, 4.1, and 4.2 are to be understood in the usual Hilbert-space sense. Note in the case $\tilde{T} = |T| U^* = T^*$ that

$$\|T + E_{A,B}(X)\|^2_2 = \|E_{A,B}(X)\|^2_2 + \|T\|^2_2$$

(5.2)

for all $X \in C_2$ if and only if $T \in \ker E_{A,B}$.

(4) Theorem 4.4 does not hold in the case $0 < p \leq 1$ because the functional calculus argument involving the function $t \mapsto t^{1/(p-1)}$, where $0 \leq t < \infty$, is only valid for $1 < p < \infty$. We ask if there is another proof where this theorem still holds in the case $0 < p < 1$. For the case $p = 1$, this theorem still holds see [12, Theorem 2.3].

6. On minimizing $\|T - (AXB - X)\|_p^p$. Maher [9, Theorem 3.2] showed that, if $A$ is normal, $AT = TA$, $1 \leq p < \infty$, and $S \in \ker \delta_{A,B} \cap C_p$; then the map $F_p$ defined by $F_p(X) = \|S - (AX - XA)\|_p^p$ has a global minimizer at $V$ if, and for $1 < p < \infty$ only if, $AV - VA = 0$. In other words, we have

$$\|S - (AX - XA)\|_p^p \geq \|T\|_p^p$$

(6.1)

if, and for $1 < p < \infty$ only if, $AV - VA = 0$. In [10] we generalized Maher’s result, showing that if the pair $(A,B)$ has the property $(FP)_{C_p}$, that is, $(AT = TB$, where $T \in C_p$ implies $A^* T = TB^*$), $1 \leq p < \infty$ and $S \in \ker \delta_{A,B} \cap C_p$, then the map $F_p$
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defined by \( F_p(X) = \|S - (AX - XB)\|_p^p \) has a global minimizer at \( V \) if, and for
\( 1 < p < \infty \) only if, \( AV - VB = 0 \). In other words, we have

\[
\|S - (AX - XB)\|_p^p \geq \|T\|_p^p
\]  

(6.2)

if, and for \( 1 < p < \infty \) only if, \( AV - VB = 0 \). In this paper, we obtain an inequality similar to (6.1), where the operator
\( AX - XB \) is replaced by the operator
\( \Delta_{A,B}(X) = AXB - X \) (in the case \( n = 1 \)). We prove that if \( \ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*} \) and
\( T \in \ker \Delta_{A,B} \cap C_p \), then the map \( F_p \) defined by \( F_p(X) = \|T - (AXB - X)\|_p^p \)
has a global minimizer at \( V \) if, and for \( 1 < p < \infty \) only if, \( AVB - V = 0 \). In other
words, we have

\[
\|T - (AXB - X)\|_p^p \geq \|T\|_p^p
\]  

(6.3)

if, and for \( 1 < p < \infty \) only if, \( AVB - V = 0 \). Additionally, we show that if
\( \ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*} \) and \( T \in \ker \Delta_{A,B} \cap C_p \), \( 1 < p < \infty \), then the map \( F_p \) has
a critical point at \( W \) if and only if \( AWB - W = 0 \), that is, if \( \mathcal{D}_W F_p \) is the Frechet
derivative at \( W \) of \( F_p \), the set

\[
\{W \in \mathcal{B}(H) : \mathcal{D}_W F_p = 0\}
\]  

(6.4)

coinsides with \( \ker \Delta_{A,B} \) (the kernel of \( \Delta_{A,B} \)).

**Theorem 6.1** [2]. If \( 1 < p < \infty \), then the map

\[
F_p : C_p \rightarrow \mathbb{R}^+
\]  

(6.5)

defined by \( X \mapsto \|X\|_p^p \) is differentiable at every \( X \in C_p \) with derivative \( \mathcal{D}_X F_p \)
given by

\[
\mathcal{D}_X F_p(T) = p \text{Re} \text{tr} (|X|^{p-1}U^*T),
\]  

(6.6)

where \( \text{tr} \) denotes trace, \( \text{Re} z \) is the real part of a complex number \( z \), and \( X = U|X| \)
is the polar decomposition of \( X \). If \( \dim \mathcal{H} < \infty \), then the same result holds for
\( 0 < p \leq 1 \) at every invertible \( X \).

**Theorem 6.2** [6]. If \( \mathcal{U} \) is a convex set of \( C_p \) with \( 1 < p < \infty \), then the map
\( X \mapsto \|X\|_p^p \), where \( X \in \mathcal{U} \), has at most a global minimizer.

**Definition 6.3.** Let \( \mathcal{U}(A,B) = \{X \in B(H) : AXB - X \in C_p\} \) and let \( F_p : \mathcal{U} \rightarrow \mathbb{R}^+ \) be the map defined by \( F_p(X) = \|T - (AXB - X)\|_p^p \), where \( T \in \ker \Delta_{A,B} \cap C_p \)
\( (1 \leq p < \infty) \).

By a simple modification in the proof of **Lemma 4.3**, we can proof the following lemma.

**Lemma 6.4.** Let \( A,B \in B(H) \) and \( S \in B(H) \) such that \( \ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*} \). If
\( A|S|^{p-1}U^*B = |S|^{p-1}U^* \), where \( p > 1 \) and \( S = U|S| \) is the polar decomposition of \( S \), then \( A|S|U^*B = |S|U^* \).
**Theorem 6.5.** Let $A, B \in B(H)$. If $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$ and $T \in \ker \Delta_{A,B} \cap C_p$, then, for $1 \leq p < \infty$, the map $F_p$ has a global minimizer at $W$ if, and for $1 < p < \infty$ only if, $AWB - W = 0$.

**Proof.** If $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, then it follows from Theorem 3.5 that $\|T - (AXB - X)\|_p^p \geq \|T\|_p^p$, that is, $F_p(X) \geq F_p(W)$. Conversely, if $F_p$ has a minimum, then

$$\|T - (AWB - WB)\|_p^p = \|S\|_p^p.$$

(6.7)

Since $\mathcal{U}$ is convex, then the set $\mathcal{V} = \{T - (AXB - X); X \in \mathcal{U}\}$ is also convex. Thus Theorem 6.2 implies that $S - (AWB - W) = S$. □

**Theorem 6.6.** Let $A, B \in B(H)$. If $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$ and $S \in \ker \Delta_{A,B} \cap C_p$, then, for $1 < p < \infty$, the map $F_p$ has a critical point at $W$ if and only if $AWB - W = 0$.

**Proof.** Let $W, S \in \mathcal{U}$ and let $\phi$ and $\varphi$ be two maps defined, respectively, by $\phi : X \mapsto S - (AXB - X)$ and $\varphi : X \mapsto \|X\|_p^p$.

Since the Frechet derivative of $F_p$ is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \to 0} \frac{F_p(W + hT) - F_p(W)}{h},$$

(6.8)

it follows that $\mathcal{D}_W F_p(T) = \mathcal{D}_{S - (AWB - W)}(ATB - T)$. If $W$ is a critical point of $F_p$, then $\mathcal{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying Theorem 6.1, we get

$$\mathcal{D}_W F_p(T) = p \text{ Retr} \left[ (S - (AWB - W) \right]^{p-1} W^* (ATB - T) \left]^{p-1} W^* \right) = p \text{ Retr} \left[ Y(ATB - T) \right] = 0,$$

(6.9)

where $S - (AWB - W) = W|S - (AWB - W)|$ is the polar decomposition of the operator $S - (AWB - W)$, and $Y = |S - (AWB - W)|^{p-1} W^*$. An easy calculation shows that $AYB - Y = 0$, that is,

$$A|S - (AWB - W)|^{p-1} W^* B = |S - (AWB - W)|^{p-1} W^*.$$

(6.10)

It follows from Lemma 6.4 that

$$A|S - (AWB - W)| W^* B = |S - (AWB - W)| W^*.$$

(6.11)

By taking adjoints and since $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, we get $A(T - (AWB - W))B = (T - (AWB - W))$. Then $A(AWB - W)B = (AWB - W)$.

Hence $AWB - W \in R(\Delta_{A,B}) \cap \ker \Delta_{A,B}$. It is easy to see that (arguing as in the proof of Theorem 3.5) if $A, B \in \mathcal{B}(H)$, $\ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*}$, and $T \in \ker \Delta_{A,B}$,
where \( T \in \mathcal{B}(H) \), then

\[
\| T - (AXB - X) \| \geq \| T \| \tag{6.12}
\]

holds for all \( X \in \mathcal{B}(H) \) and for all \( T \in \ker \Delta_{A,B} \). Hence \( AWB - W = 0 \).

Conversely, if \( AWB = W \), then \( W \) is a minimum, and since \( F_p \) is differentiable, then \( W \) is a critical point.

**Theorem 6.7.** Let \( A, B \in \mathcal{B}(H) \) such that \( \ker \Delta_{A,B} \subseteq \ker \Delta_{A^*,B^*} \), \( S \in \ker \Delta_{A,B} \cap C_p \) \((0 < p \leq 1)\), \( \dim \mathfrak{H} < \infty \), and \( S - (AWB - W) \) is invertible. Then \( F_p \) has a critical point at \( W \) if \( AWB - W = 0 \).

**Proof.** Suppose that \( \dim \mathfrak{H} < \infty \). If \( AWB - W = 0 \), then \( S \) is invertible by hypothesis. Also \( |S| \) is invertible, hence \( |S|^{p-1} \) exists for \( 0 < p \leq 1 \). If we take

\[
Y = |S|^{p-1}U^* \tag{6.13}
\]

with \( S = U|S| \) the polar decomposition and since \( ASB = S \) implies \( BS^*A = S^* \), then \( AS^*S = AS^*BSA = S^*SA \), and this implies that \( |S|^2A = A|S|^2 \) and \( |S|A = A|S| \).

Since \( BS^*A = S^* \), that is, \( A|S|U^*B = |S|U^*, |S|(AU^*B - U^*) = 0 \), and since \( A|S|^{p-1} = |S|^{p-1}A \), then

\[
AYB - Y = A|S|^{p-1}U^*B - |S|^{p-1}U^* = |S|^{p-1}(AU^*B - U^*) \tag{6.14}
\]

so that \( AYB - Y = 0 \) and \( \text{tr}(AYB - Y) = 0 \) for all \( T \in B(H) \). Since \( S = S - (AWB - W) \), then

\[
0 = \text{tr}[YATB - YAT] = \text{tr}[Y(ATB - T)]
= p \text{Re} \text{tr} [Y(ATB - AT)] = p \text{Re} \text{tr} [|S|^{p-1}U^*(ATB - T)]
= (\mathfrak{S}_T \phi)(ATB - T) = (\mathfrak{S}_W F_p)(T). \tag{6.15}
\]

**Remark 6.8.** (1) In Theorem 6.6, the implication “\( W \) is a critical point \( \Rightarrow AWB - WB = 0 \)” does not hold in the case \( 0 < p \leq 1 \) because the functional calculus argument involving the function \( t \mapsto t^{1/(p-1)} \), where \( 0 \leq t < \infty \), is only valid for \( 1 < p < \infty \).

(2) Theorems 3.5, 6.5, 6.6, and 6.7 hold in particular if \( A \) and \( B \) are contractions. Indeed, it is known [4] that if \( A \) and \( B \) are contractions and \( \Delta_{A,B}(S) = 0 \), where \( S \in C_p \), then \( \Delta_{A^*,B^*}(S) = \delta_{A^*,B}(S) = \delta_{A,B^*}(S) = 0 \).

(3) The set

\[
\mathcal{F} = \{ X : AXB - X \in C_p \} \tag{6.16}
\]

contains \( C_p \) for if \( X \in C_p \), then \( X \in \mathcal{F} \) and, for example, \( I \in \mathcal{F} \) but \( I \notin C_p \). If \( A \in C_p \), the conclusions of Theorems 6.5, 6.6, and 6.7 hold for all \( X \in B(H) \).
7. On minimizing \( \| T - (\sum_{i=1}^{n} A_i X B_i - X) \|_p^p \). Let \( A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) be \( n \)-tuples of operators in \( B(H) \). We define the elementary operator \( E_{A,B} : B(H) \to B(H) \) by \( E_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X \).

Denote \( E_{A,A} = E_A \). In this section, we prove that if \( \sum_{i=1}^{n} A_i A_i^* \leq 1, \sum_{i=1}^{n} A_i^* A_i \leq 1, \sum_{i=1}^{n} B_i B_i^* \leq 1, \sum_{i=1}^{n} B_i^* B_i \leq 1, \ker E_{A,B} \subseteq \ker E_{A^*,B^*} \), and \( T \in \ker \Delta_{A,B} \cap \ker E_c \), then the map \( F_p \) defined by \( F_p(X) = \| T - E_{A,B}(X) \|_p^p \) has a global minimizer at \( V \) if, and for \( 1 < p < \infty \) only if, \( \sum_{i=1}^{n} A_i V B_i - V = 0 \). In other words, we have

\[
\| T - E_{A,B}(X) \|_p^p \geq \| T \|_p^p \tag{7.1}
\]

if, and for \( 1 < p < \infty \) only if, \( \sum_{i=1}^{n} A_i V B_i - V = 0 \). Additionally, we show that if \( \ker E_{A,B} \subseteq E_{A^*,B^*} \) and \( T \in \ker E_{A,B} \cap \ker E_c \) \( (1 < p < \infty) \), then the map \( F_p \) has a critical point at \( W \) if and only if \( \sum_{i=1}^{n} A_i W B_i - W = 0 \), that is, if \( D_W F_p \) is the Fréchet derivative of \( F_p \) at \( W \), the set

\[
\{ W \in L(H) : D_W F_p = 0 \} \tag{7.2}
\]

coincides with \( \ker E_{A,B} \) (the kernel of \( E_{A,B} \)).

**Definition 7.1.** Let \( \mathcal{U}(A,B) = \{ X \in B(H) : (\sum_{i=1}^{n} C_i X C_i - X) \in \ker E_c \} \) and let \( F_p : \mathcal{U} \to \mathbb{R}^+ \) be the map defined by \( F_p(X) = \| T - (\sum_{i=1}^{n} C_i X C_i - X) \|_p^p \), where \( T \in \ker E_C \cap \ker E_c \) \((1 \leq p < \infty)\).

**Lemma 7.2.** Let \( C = (C_1, C_2, \ldots, C_n) \) be \( n \)-tuple of operators in \( B(H) \) such that \( \sum_{i=1}^{n} C_i C_i^* \leq 1, \sum_{i=1}^{n} C_i^* C_i \leq 1, \) and \( \ker E_C \subseteq \ker E^*_C \). If \( \sum_{i=1}^{n} |C_i S|^{p-1} U^* C_i = |S|^{p-1} U^* \), where \( p > 1 \) and \( S = U[S] \) is the polar decomposition of \( S \), then \( \sum_{i=1}^{n} C_i S|U^* C_i = |S| U^* \).

**Proof.** By the same arguments as in the proof of Lemma 4.3, the proof can be completed.

**Theorem 7.3.** Let \( C = (C_1, C_2, \ldots, C_n) \) be \( n \)-tuple of operators in \( B(H) \). If \( \sum_{i=1}^{n} C_i C_i^* \leq 1, \sum_{i=1}^{n} C_i^* C_i \leq 1, \) and \( \ker E_C \subseteq \ker E^*_C \), and \( T \in \ker \Delta_{A,B} \cap \ker E_c \), then, for \( 1 \leq p < \infty \), the map \( F_p \) has a global minimizer at \( W \) if, and for \( 1 < p < \infty \) only if, \( \sum_{i=1}^{n} C_i W C_i - W = 0 \).

**Proof.** If \( \sum_{i=1}^{n} C_i C_i^* \leq 1, \sum_{i=1}^{n} C_i^* C_i \leq 1, \) and \( \ker E_C \subseteq \ker E^*_C \), it follows from Theorem 4.4 that

\[
\left\| T - \left( \sum_{i=1}^{n} C_i X C_i - X \right) \right\|_p^p \geq \| T \|_p^p, \tag{7.3}
\]

that is, \( F_p(X) \geq F_p(W) \). Conversely, if \( F_p \) has a minimum, then

\[
\left\| T - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right\|_p^p = \| T \|_p^p. \tag{7.4}
\]
Since \( \mathcal{U} \) is convex, then the set
\[
\mathcal{V} = \left\{ T - \left( \sum_{i=1}^{n} C_i X C_i - X \right) ; X \in \mathcal{U} \right\}
\]
(7.5)
is also convex. Thus Theorem 6.2 implies that \( T - (\sum_{i=1}^{n} C_i W C_i - W) = T \). \( \Box \)

**Theorem 7.4.** Let \( C = (C_1, C_2, \ldots, C_n) \) be an \( n \)-tuple of operators in \( B(H) \). If \( \sum_{i=1}^{n} C_i C_i^* \leq 1, \sum_{i=1}^{n} C_i^* C_i \leq 1, \ker E_c \subseteq \ker E_c^*, \) and \( T \in \ker E_c \cap C_p \), then, for \( 1 \leq p < \infty \), the map \( F_p \) has a critical point at \( W \) if, and for \( 1 < p < \infty \) only if,
\[
\sum_{i=1}^{n} C_i W C_i - W = 0.
\]
(7.6)

**Proof.** Let \( W, S \in U \) and let \( \phi \) and \( \varphi \) be two maps defined, respectively, by
\[
\phi : X \mapsto S - \left( \sum_{i=1}^{n} C_i X C_i - X \right), \quad \varphi : X \mapsto \|X\|_p^p.
\]
(7.7)

Since the Frechet derivative of \( F_p \) is given by
\[
\mathcal{D}_W F_p(T) = \lim_{h \to 0} \frac{F_p(W + hT) - F_p(W)}{h},
\]
(7.8)
it follows that
\[
\mathcal{D}_W F_p(T) = \left[ \mathcal{D}_{S - (\sum_{i=1}^{n} C_i W C_i - W)} \right] \left( \sum_{i=1}^{n} C_i T C_i - T \right).
\]
(7.9)

If \( W \) is a critical point of \( F_p \), then \( \mathcal{D}_W F_p(T) = 0 \) for all \( T \in \mathcal{U} \). By applying Theorem 6.1, we get
\[
\mathcal{D}_W F_p(T) = p \text{Re} \text{tr} \left[ \left| S - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right|^{p-1} W^* \left( \sum_{i=1}^{n} C_i T C_i - T \right) \right]
\]
(7.10)
where
\[
S - \left( \sum_{i=1}^{n} C_i W C_i - W \right) = W \left( \sum_{i=1}^{n} C_i W C_i - W \right),
\]
(7.11)
is the polar decomposition of the operator \( S - (\sum_{i=1}^{n} C_i W C_i - W) \), and
\[
Y = \left| S - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right|^{p-1} W^*.
\]
(7.12)
An easy calculation shows that
\[
\left( \sum_{i=1}^{n} C_i Y C_i - Y \right) = 0,
\]
that is,
\[
\sum_{i=1}^{n} C_i \left| S - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right|^{p-1} W^* C_i = \left| S - (AWB - W) \right|^{p-1} W^*.
\] (7.13)

It follows from Lemma 7.2 that
\[
\sum_{i=1}^{n} C_i \left| S - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right| W^* C_i = \left| S - (AWB - W) \right| W^*.
\] (7.14)

By taking adjoints, and since \( \ker E_C \subseteq \ker E_{C^*} \), we get
\[
\sum_{i=1}^{n} C_i \left( T - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right) C_i = \left( T - \left( \sum_{i=1}^{n} C_i W C_i - W \right) \right),
\] (7.15)
and then
\[
\sum_{i=1}^{n} C_i \left( \sum_{i=1}^{n} C_i W C_i - W \right) C_i = \left( \sum_{i=1}^{n} C_i W C_i - W \right).
\] (7.16)
Hence
\[
\sum_{i=1}^{n} C_i W C_i - W \in R(E_C) \cap \ker E_{C^*}.
\] (7.17)

It is easy to see that (arguing as in the proof of [14, Proposition 4.3]) if \( C = (C_1, C_2, \ldots, C_n) \) is \( n \)-tuple of operator in \( B(H) \) such that
\[
\sum_{i=1}^{n} C_i C_i^* \leq 1, \quad \sum_{i=1}^{n} C_i^* C_i \leq 1,
\]
\( \ker E_c \subseteq \ker E_{c^*} \), and \( T \in \ker \Delta_C \), where \( T \in B(H) \), then
\[
\| T - \Delta_C X \| \geq \| T \|
\] (7.20)
holds for all \( X \in B(H) \) and for all \( T \in \ker E_c \). Hence \( \sum_{i=1}^{n} C_i W C_i - W = 0 \).

Conversely, if \( \sum_{i=1}^{n} C_i W C_i = W \), then \( W \) is minimum, and since \( F_p \) is differentiable, \( W \) is a critical point.

**Theorem 7.5.** Let \( C = (C_1, C_2, \ldots, C_n) \) be \( n \)-tuple of operators in \( B(H) \). If
\[
\sum_{i=1}^{n} C_i C_i^* \leq 1, \quad \sum_{i=1}^{n} C_i^* C_i \leq 1,\] (7.21)
such that \( \ker E_c \subseteq \ker E_c^* \), \( S \in \ker E_c \cap C_p \) (\( 0 < p \leq 1 \)), \( \dim H < \infty \), and \( S - (\sum_{i=1}^{n} C_i W C_i - W) \) is invertible, then \( F_p \) has a critical point at \( W \) if \( \sum_{i=1}^{n} C_i W C_i - W = 0 \).

**Proof.** Suppose that \( \dim H < \infty \). If \( \sum_{i=1}^{n} C_i W C_i - W = 0 \), then \( S \) is invertible by hypothesis. Also \( |S| \) is invertible, hence \( |S|^{p-1} \) exists for \( 0 < p \leq 1 \) taking \( Y = |S|^{p-1} U^* \), where \( S = U|S| \) is the polar decomposition of \( S \).

It is known that if

\[
\sum_{i=1}^{n} C_i C_i^* \leq 1, \quad \sum_{i=1}^{n} C_i^* C_i \leq 1, \quad \ker E_c \subseteq \ker E_c^*,
\]

(7.22)

the eigenspaces corresponding to distinct nonzero eigenvalues of the compact positive operator \( |S|^2 \) reduce each \( C_i \) (see [5, Theorem 8] and [14, Lemma 2.3]). In particular, \( |S| \) commutes with \( C_i \) for all \( 1 \leq i \leq n \). Hence

\[
C_i |S| = |S| C_i.
\]

(7.23)

Since \( \sum_{i=1}^{n} C_i S^* C_i = S^* \), that is,

\[
\sum_{i=1}^{n} C_i |S| U^* C_i = |S| U^*,
\]

(7.24)

then

\[
|S| \left( \sum_{i=1}^{n} C_i U^* C_i - U^* \right) = 0,
\]

(7.25)

and since

\[
A |S|^{p-1} = |S|^{p-1} A,
\]

(7.26)

then

\[
\sum_{i=1}^{n} C_i Y C_i - Y = \sum_{i=1}^{n} C_i |S|^{p-1} U^* C_i - |S|^{p-1} U^* = |S|^{p-1} \left( \sum_{i=1}^{n} C_i U^* C_i - U^* \right)
\]

(7.27)

so that \( \sum_{i=1}^{n} C_i Y C_i - Y = 0 \) and \( \text{tr}[\left( \sum_{i=1}^{n} C_i Y C_i - Y \right) T] = 0 \) for all \( T \in B(H) \). Since

\[
S = S - \left( \sum_{i=1}^{n} C_i W C_i - W \right),
\]

(7.28)
then
\[ 0 = \text{tr} \left[ Y \sum_{i=1}^{n} C_i TC_i - YT \right] = \text{tr} \left[ Y \left( \sum_{i=1}^{n} C_i TC_i - T \right) \right] = \text{tr} \left[ Y \left( \sum_{i=1}^{n} C_i TC_i - T \right) \right] = p \text{Retr} \left[ Y \left( \sum_{i=1}^{n} C_i TC_i - T \right) \right] = p \text{Retr} \left[ |S|^{p-1} U^* \left( \sum_{i=1}^{n} C_i TC_i - T \right) \right] \]
\[ = (\mathcal{Z}_T \phi) \left( \sum_{i=1}^{n} C_i TC_i - T \right) = (\mathcal{Z}_W F_p)(T). \] (7.29)

**Theorem 7.6.** Let \( A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) be \( n \)-tuples of operators in \( B(H) \) such that
\[ \sum_{i=1}^{n} A_i A_i^* \leq 1, \quad \sum_{i=1}^{n} A_i^* A_i \leq 1, \quad \sum_{i=1}^{n} B_i B_i^* \leq 1, \quad \sum_{i=1}^{n} B_i^* B_i \leq 1. \] (7.30)

If
\[ \ker E_{A,B} \subseteq \ker E_{A^*,B^*} \] (7.31)
and \( T \in \ker E_{A,B} \cap C_p \), then for \( 1 \leq p < \infty \),

(i) the map \( F_p \) has a global minimizer at \( W \) if, and for \( 1 < p < \infty \) only if,
\[ \sum_{i=1}^{n} A_i WB_i - W = 0; \]
(ii) the map \( F_p \) has a critical point at \( W \) if, and for \( 1 < p < \infty \) only if,
\[ \sum_{i=1}^{n} A_i WB_i - W = 0; \]
(iii) for \( 0 < p \leq 1 \), \( \dim H < \infty \), and \( S - (\sum_{i=1}^{n} C_i WC_i - W) \) invertible, \( F_p \) has a critical point at \( W \) if
\[ \sum_{i=1}^{n} A_i WB_i - W = 0. \]

**Proof.** It suffices to take the Hilbert space \( H \oplus H \) and operators (4.10) and apply Theorems 7.3, 7.4, and 7.5. \( \square \)

**Remark 7.7.** (1) In Theorem 7.4, the implication “\( W \) is a critical point \( \Rightarrow \sum_{i=1}^{n} A_i WB_i - W = 0 \)” does not hold in the case \( 0 < p \leq 1 \) because the functional calculus argument involving the function \( t \rightarrow t^{1/(p-1)} \), where \( 0 \leq t < \infty \), is only valid for \( 1 < p < \infty \).

(2) The set \( S = \{ X : A XB - X \in C_p \} \) contains \( C_p \). If \( X \in C_p \), then \( X \in S \) and, for example, \( I \in S \) but \( I \notin C_p \). If \( A \in C_p \), the conclusion of Theorem 7.6 holds for all \( X \in B(H) \).
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References


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