REDUCTIVE COMPACTIFICATIONS OF SEMITOPOLOGICAL SEMIGROUPS

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We consider the enveloping semigroup of a flow generated by the action of a semitopological semigroup on any of its semigroup compactifications and explore the possibility of its being one of the known semigroup compactifications again. In this way, we introduce the notion of $E$-algebra, and show that this notion is closely related to the reductivity of the semigroup compactification involved. Moreover, the structure of the universal $E\mathcal{F}$-compactification is also given.

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1. Introduction. A semigroup $S$ is called right reductive if $a = b$ for each $a, b \in S$, since $at = bt$ for every $t \in S$. For example, all right cancellative semigroups and semigroups with a right identity are right reductive.

From now on, $S$ will be a semitopological semigroup, unless otherwise is stipulated. By a semigroup compactification of $S$ we mean a pair $(\psi, X)$, where $X$ is a compact Hausdorff right topological semigroup, and $\psi : S \to X$ is a continuous homomorphism with dense image such that, for each $s \in S$, the mapping $x \to \psi(s)x : X \to X$ is continuous. The $C^*$-algebra of all bounded complex-valued continuous functions on $S$ will be denoted by $\mathcal{C}(S)$. For $\mathcal{C}(S)$, the left and right translations, $L_s$ and $R_t$, are defined for each $s, t \in S$ by $(L_sf)(t) = f(st) = (R_tf)(s)$, $f \in \mathcal{C}(S)$. The subset $\mathcal{T}$ of $\mathcal{C}(S)$ is said to be left translation invariant if for all $s \in S$, $L_s\mathcal{T} \subseteq \mathcal{T}$. A left translation invariant unital $C^*$-subalgebra $T$ of $\mathcal{C}(S)$ is called $m$-admissible if the function $s \to T_{\mu}(f(s)) = \mu(L_sf)$ is in $T$ for all $f \in T$ and $\mu \in S^\mathcal{T}$ (where $S^\mathcal{T}$ is the spectrum of $T$). Then the product of $\mu, \nu \in S^\mathcal{T}$ can be defined by $\mu \nu = \mu \circ T_{\nu}$ and the Gelfand topology on $S^\mathcal{T}$ makes $(\mathcal{C}(S), S^\mathcal{T})$ a semigroup compactification (called the $\mathcal{T}$-compactification) of $S$, where $\epsilon : S \to S^\mathcal{T}$ is the evaluation mapping.

Some $m$-admissible subalgebras of $\mathcal{C}(S)$, that we will need, are left multiplicatively continuous functions $L\mathcal{M}\mathcal{C}$, distal functions $\mathcal{D}$, minimal distal functions $\mathcal{M}\mathcal{D}$, and strongly distal functions $\mathcal{P}\mathcal{D}$. We also write $\mathcal{G}\mathcal{P}$ for $\mathcal{M}\mathcal{D} \cap \mathcal{P}\mathcal{D}$; and we define $\mathcal{L}\mathcal{P}\mathcal{E} := \{f \in \mathcal{C}(S); f(st) = f(s) \text{ for all } s, t \in S\}$. For a discussion of the universal property of the corresponding compactifications of these function algebras see [1, 2].
2. Reductive compactifications and E-algebras. Let \((\psi, X)\) be a compactification of \(S\), then the mapping \(\sigma : S \times X \to X\), defined by \(\sigma(s, x) = \psi(s)x\), is separately continuous and so \((S, X, \sigma)\) is a flow. If \(\Sigma_X\) denotes the enveloping semigroup of the flow \((S, X, \sigma)\) (i.e., the pointwise closure of semigroup \(\{\sigma(s, \cdot) : s \in S\}\) in \(X^X\)) and the mapping \(\sigma_X : S \to \Sigma_X\) is defined by \(\sigma_X(s) = \sigma(s, \cdot)\) for all \(s \in S\), then \((\sigma_X, \Sigma_X)\) is a compactification of \(S\) (see [1, Proposition 1.6.5]).

One can easily verify that \(\Sigma_X = \{\lambda_x : x \in X\}\), where \(\lambda_x(y) = xy\) for each \(y \in X\). If we define the mapping \(\theta : X \to \Sigma_X\) by \(\theta(x) = \lambda_x\), then \(\theta\) is a continuous homomorphism with the property that \(\theta \circ \psi = \sigma_X\). So \((\sigma_X, \Sigma_X)\) is a factor of \((\psi, X)\), that is \((\psi, X) \geq (\sigma_X, \Sigma_X)\). By definition, \(\theta\) is one-to-one if and only if \(X\) is right reductive. So we get the next proposition, which is an extension of the Lawson’s result [3, Lemma 2.4(ii)].

**Proposition 2.1.** Let \((\psi, X)\) be a compactification of \(S\). Then \((\sigma_X, \Sigma_X) \equiv (\psi, X)\) if and only if \(X\) is right reductive.

A compactification \((\psi, X)\) is called reductive if \(X\) is right reductive. For example, the \(\mathcal{M}\mathcal{F}\mathcal{P}\), \(\mathcal{G}\mathcal{P}\), and \(\mathcal{L}\mathcal{F}\mathcal{P}\)-compactifications are reductive.

An \(m\)-admissible subalgebra \(\mathcal{F}\) of \(\epsilon(S)\) is called an \(E\)-algebra if there is a compactification \((\psi, X)\) such that \((\sigma_X, \Sigma_X) \equiv (\epsilon, S^\mathcal{F})\). In this setting \((\psi, X)\) is called an \(E\mathcal{F}\)-compactification of \(S\). Trivially for every reductive compactification \((\psi, X), \psi^* (\epsilon(X))\) is an \(E\)-algebra. But the converse is not, in general, true. For instance, for any compactification \((\psi, X), \sigma_X^* (\epsilon(\Sigma_X))\) is an \(E\)-algebra; however, it is possible that \(\Sigma_X\) would be nonreductive, as the next example shows.

**Example 2.2.** Let \(S = \{a, b, c, d\}\) be the semigroup with the following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<tr>
<td>a</td>
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<td>a</td>
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<td>a</td>
</tr>
<tr>
<td>b</td>
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<td>a</td>
<td>a</td>
<td>c</td>
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<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>c</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

Then for the identity compactification \((i, X)\) of \(S\), \(\Sigma_X\) is not right reductive; in fact, \(\lambda_a \neq \lambda_b\), however, \(\lambda_{at} = \lambda_{bt}\) for every \(t \in S\).

**Lemma 2.3.** If \((\psi, X)\) is a compactification satisfying \(X^2 = X\), then the compactification \((\sigma_X, \Sigma_X)\) is reductive.

**Proof.** Since \(X^2 = X\), for each \(x_1, x_2 \in X\), from \(\lambda_{x_1} \lambda_y = \lambda_{x_2} \lambda_y\) for every \(\lambda_y \in \Sigma_X\), it follows that \(\lambda_{x_1} \lambda_{x_2} \lambda_y\). So \(\Sigma_X\) is right reductive. \(\square\)

**Corollary 2.4.** Let \(sS\) (or \(Ss\)) be dense in \(S\), for some \(s \in S\), then for every compactification \((\psi, X)\) of \(S\), it follows that \(X^2 = X\) and so \((\sigma_X, \Sigma_X)\) is reductive.
Now, we are going to construct the universal $E\mathcal{F}$-compactification of $S$. For this end we need the following lemma.

**Lemma 2.5.** Let $\mathcal{F}$ be an $m$-admissible subalgebra of $\mathcal{C}(S)$. Then $T_\nu f \in \sigma_\delta^\mu (\mathcal{C}(\Sigma_\delta^\mu))$ for all $f \in \mathcal{F}$ and $\nu \in S^{2,\mu,\nu,\eta}$. 

**Proof.** Since $\Sigma_\delta^\mu = \{\lambda_\mu : \mu \in S^\delta\}$, we can define $g : \Sigma_\delta^\mu \to \mathbb{C}$ by $g(\lambda_\mu) = \mu(T_\nu f)$, where $\mathbb{C}$ denotes the complex numbers. Since the mapping $\lambda_\mu \to \mu \nu : \Sigma_\delta^\mu \to S^\delta$ is $p$-weak* continuous, $g$ is a bounded continuous function and it is easy to see that $\sigma_\delta^\mu (g) = T_\nu (f)$. Therefore, $T_\nu f \in \sigma_\delta^\mu (\mathcal{C}(\Sigma_\delta^\mu))$ for all $\nu \in S^\delta$. If $\nu \in S^{2,\mu,\nu,\eta}$, then $T_\nu f = f$ for all $f \in \mathcal{F}$. So the conclusion follows.

**Proposition 2.6.** Let $\mathcal{F}$ be an $E$-algebra. Then

$$G_\mathcal{F} := \{f \in L\mathcal{F}(\mu) : T_\nu f \in \mathcal{F} \ \forall \nu \in S^{2,\mu,\nu,\eta}\}$$

is an $m$-admissible subalgebra of $\mathcal{C}(S)$ and $(\mathcal{C}(S), \Sigma_\delta^\mu)$ is the universal $E\mathcal{F}$-compactification of $S$.

**Proof.** It is easy to verify that $G_\mathcal{F}$ is an $m$-admissible subalgebra of $\mathcal{C}(S)$ containing $\mathcal{F}$. By definition of $G_\mathcal{F}$ we can define the mapping $\theta : S^\mu \to \Sigma_\delta^\mu$ by $\theta(\mu) = \lambda_\mu$, where $\lambda_\mu$ is an extension of $\mu$ to $S^\delta$. Clearly, $\theta$ is continuous and $\theta \circ \epsilon = \sigma_\delta^\mu$. Thus $(\mathcal{C}(S), S^\delta) \geq (\sigma_\delta^\mu, S^\delta)$. On the other hand, since $\mathcal{F}$ is an $E$-algebra, there exists a compactification $(\phi, Y)$ of $S$ such that $(\sigma_\delta^\mu, S^\delta) \geq (\epsilon, S^\delta)$ and $\mathcal{F} = \sigma_\delta^\mu (\mathcal{C}(\Sigma_\delta^\mu))$. By Lemma 2.5, we have $T_\nu f \in \sigma_\delta^\mu (\mathcal{C}(\Sigma_\delta^\mu))$, for each $\nu \in S^{2,\mu,\nu,\eta}$ and each $f \in \mathcal{F}$, so by [1, Proposition 1.6.7], $(\sigma_\delta^\mu, S^\delta) \geq (\sigma_\delta^\mu, S^\delta)$. Therefore, $(\mathcal{C}(S), S^\delta) \geq (\sigma_\delta^\mu, S^\delta)$ and $(\mathcal{C}(S), S^\delta)$ is an $E\mathcal{F}$-compactification of $S$. Finally, if $(\psi, X)$ is an $E\mathcal{F}$-compactification of $S$ and $f \in \mathcal{F}(\psi, X)$, then by Lemma 2.5, $T_\nu f \in \sigma_\delta^\mu (\mathcal{C}(\Sigma_\delta^\mu))$ for all $\mu \in S^{2,\mu,\nu,\eta}$. So $\psi (\mathcal{F}(\psi, X)) \subset G_\mathcal{F}$ and $(\psi, X) \leq (\mathcal{C}(S), S^\delta)$. 

**Examples 2.7.**

(a) We have $G_{\mathcal{D}} = \mathcal{D}$. To see this, if $f \in G_{\mathcal{D}}$, then for all $\mu, \nu, \eta \in S^{2,\mu,\nu,\eta}$ with $\eta^2 = \eta$, we have $\mu \nu = \mu(T_\nu f) = \mu(T_\nu f) = \nu(T_\nu f)$. So $f \in \mathcal{D}$. Also if $f \in \mathcal{D}$, then for all $\mu, \nu, \eta \in S^{2,\mu,\nu,\eta}$ with $\eta^2 = \eta$, we have $\mu \eta(T_\nu f) = \mu \eta(T_\nu f) = \mu(T_\nu f)$. That is, $T_\nu f \in \mathcal{D}$ for all $\nu \in S^{2,\mu,\nu,\eta}$ and so $f \in G_{\mathcal{D}}$ (see also [4, Lemma 2.2]).

(b) By a similar proof, we can show that $G_{\mathcal{D}} = \mathcal{D}$ (see [4, Lemma 2.2 and Theorem 2.6]).

(c) Let $\mathcal{R} := \{f \in L\mathcal{M}(S) : f(rst) = f(rt) \text{ for } r, s, t \in S\}$. Clearly, $\mathcal{R}$ is an $m$-admissible subalgebra of $\mathcal{C}(S)$. If $f \in \mathcal{R}$ and $\nu \in S^{2,\mu,\nu,\eta}$, then for each $r, s, t \in S$ we have $L_{rst} f(s) = f(rts) = f(rts) = L_{rst} f(s)$. So $T_\nu f(rt) = \nu(L_{rst} f) = \nu(L_{rst} f) = T_\nu f(r)$. That is, $T_\nu f \in \mathcal{R}$. On the other hand, if $f \in G_{\mathcal{D}}$, then $f(rst) = (T_{\epsilon(t)} f) (rs) = (T_{\epsilon(t)} f) (r) = f(rt)$ and so $f \in \mathcal{R}$. Therefore, $G_{\mathcal{D}} = \mathcal{R}$. 

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