PARA-$f$-LIE GROUPS

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Special para-$f$-structures on Lie groups are studied. It is shown that every para-$f$-Lie group $G$ is the quotient of the product of an almost product Lie group and a Lie group with trivial para-$f$-structure by a discrete subgroup.

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1. Para-$f$-structures. The notion of a para-$f$-structure on a differentiable manifold was introduced and studied in [2].

Let $M$ be an $n$-dimensional differentiable manifold of class $C^\infty$. The set of all vector fields on $M$ will be denoted by $\mathfrak{X}(M)$ and the tangent space of $M$ at a point $m \in M$ by $T_mM$.

**Definition 1.1.** Let $M$ be an $n$-dimensional differentiable manifold. If $\varphi$ is an endomorphism field of constant rank $k$ on $M$ satisfying

$$\varphi^3 - \varphi = 0,$$  

then $\varphi$ is called a **para-$f$-structure** on $M$ and $M$ is a para-$f$-manifold.

**Definition 1.2.** A para-$f$-structure $\varphi$ on $M$ is **integrable** if there exists a coordinate system in which $\varphi$ has constant components

$$\begin{bmatrix}
I_p & 0 & 0 \\
0 & -I_q & 0 \\
0 & 0 & 0
\end{bmatrix},$$

where $I$ is the unit matrix and $p + q = k$.

**Proposition 1.3.** A para-$f$-structure $\varphi$ on $M$ is integrable if and only if its Nijenhuis tensor field $N_\varphi$ vanishes, that is,

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] + \varphi^2[X,Y] = 0,$$

where $X,Y \in \mathfrak{X}(M)$. 

For a para-$f$-structure $\varphi$ on $M$, let
\begin{align}
\ker \varphi &= \bigcup_{m \in M} (\ker \varphi)_m, \\
\im \varphi &= \bigcup_{m \in M} (\im \varphi)_m
\end{align}
(1.4)
be the kernel and image of $\varphi$, respectively, where
\begin{align}
(\ker \varphi)_m &= \{X \in T_mM; \varphi_m(X) = 0\}, \\
(\im \varphi)_m &= \{Y \in T_mM; Y = \varphi_m(X) \text{ for some } X \in T_mM\}
\end{align}
(1.5)
are the kernel and image of $\varphi$ at any point $m \in M$, respectively.

**Proposition 1.4.** If $(\ker \varphi)_m = \{0\}$ for a para-$f$-structure $\varphi$ for all $m \in M$, then $\varphi$ is an almost product structure on $M$, that is, $\varphi^2 = \Id$.

**Proposition 1.5.** If $(\im \varphi)_m = \{0\}$ for a para-$f$-structure $\varphi$ for all $m \in M$, then $\varphi$ is the trivial para-$f$-structure on $M$, that is, $\varphi = 0$.

**Proposition 1.6.** If $\varphi$ is a para-$f$-structure on $M$, then
\[ \ker \varphi \cap \im \varphi = \{0\}. \]
(1.6)

**Proof.** If $Z \in \ker \varphi \cap \im \varphi$, then $\varphi(Z) = 0$, and there exists $X$ such that $\varphi(X) = Z$. Hence $\varphi^2(X) = 0$, and from Definition 1.1, we get $0 = \varphi^3(X) = \varphi(X) = Z$. \hfill $\Box$

**Definition 1.7.** Let $\varphi_i$ be a para-$f$-structure on a para-$f$-manifold $M_i$ with $i = 1, 2$. A diffeomorphism $h : M_1 \to M_2$ is called a para-$f$-map if
\[ \varphi_2 \circ h_\ast = h_\ast \circ \varphi_1, \]
(1.7)
where $h_\ast$ is the differential of $h$.

2. Para-$f$-Lie groups. In this section, the notion of a para-$f$-Lie group is introduced. Some properties of its Lie algebra are established. Finally, its special decomposition in terms of an almost product Lie group and a Lie group with trivial para-$f$-structure is proved.

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. As usual, we define
\begin{align}
L_g : G &\to G \quad \text{(left multiplication by } g \in G), \\
R_g : G &\to G \quad \text{(right multiplication by } g \in G), \\
ad_g : G &\to G, \quad a \mapsto ad_g(a) = gag^{-1}, \\
\Ad_X : \mathfrak{g} &\to \mathfrak{g}, \quad Y \mapsto \Ad_X(Y) = [X,Y].
\end{align}
(2.1)

**Definition 2.1.** Let $G$ be a Lie group with a para-$f$-structure $\varphi$. If both $L_g$ and $R_g$ are para-$f$-maps, then $\varphi$ is said to be bi-invariant.
**Definition 2.2.** If \( G \) is a Lie group with an integrable bi-invariant para-f-structure \( \varphi \), then \( G \) is called a *para-f-Lie* group.

**Proposition 2.3.** If \( \varphi \) is a bi-invariant para-f-structure on a Lie group \( G \), then

\[
\varphi[X,Y] = [\varphi(X),Y]
\]

(2.2)

for all \( X,Y \in \mathfrak{g} \).

**Proof.** Since \( \varphi \circ (L_g)_* = (L_g)_* \circ \varphi \) and \( \varphi \circ (R_g)_* = (R_g)_* \circ \varphi \), we have \( \varphi \circ (\text{ad}_g)_* = (\text{ad}_g)_* \circ \varphi \) for all \( g \in G \). If \( g = \exp(tX) \), where \( t \in \mathbb{R} \), then \( \varphi \circ (\text{ad}_{\exp(tX)})_* = (\text{ad}_{\exp(tX)})_* \circ \varphi \). Hence, by a standard result in Lie groups,

\[
\varphi \circ e^{\text{Ad}_{tX}} = e^{\text{Ad}_{tX}} \circ \varphi,
\]

(2.3)

or, for any \( Y \in \mathfrak{g} \),

\[
\varphi\left(Y + t[X,Y] + \frac{t^2}{2!}[X,[X,Y]] + \cdots\right) = \varphi(Y) + t[X,\varphi(Y)] + \frac{t^2}{2!}[X,[X,\varphi(Y)]] + \cdots.
\]

(2.4)

Hence,

\[
\varphi[X,Y] + \frac{t}{2!}[X,[X,Y]] + \cdots = [X,\varphi(Y)] + \frac{t}{2!}[X,[X,\varphi(Y)]] + \cdots.
\]

(2.5)

Letting \( t \to 0 \) in (2.5) gives us the desired result.

**Proposition 2.4.** A bi-invariant para-f-structure \( \varphi \) on a Lie group \( G \) is integrable.

**Proof.** From Proposition 2.3, the Nijenhuis tensor of a bi-invariant para-f-structure \( \varphi \) must vanish at the unity \( e \) of \( G \).

**Corollary 2.5.** A Lie group \( G \) with a bi-invariant para-f-structure \( \varphi \) is a para-f-Lie group.

**Example 2.6.** Let \( G = \text{GL}(n,\mathbb{R}) \) be the group of all real nonsingular \( n \times n \) matrices. Let \( \varphi : G \to G, \ X \mapsto \varphi(X) = X - (1/n) \text{trace}(X)I \), where \( I \) is the unit matrix. Then \( \varphi \) is a bi-invariant para-f-structure on \( G \).

**Proposition 2.7.** Let \( G \) be a para-f-Lie group with a para-f-structure \( \varphi \). Then its Lie algebra \( \mathfrak{g} \) is expressed as

\[
\mathfrak{g} = V_k \oplus V_i,
\]

(2.6)

the direct sum (as a Lie algebra), where \( V_k = (\ker \varphi)_e \) and \( V_i = (\text{im} \varphi)_e \) are subalgebras of \( \mathfrak{g} \), and \( e \in G \) is the unity of \( G \).
Proof. From Proposition 1.6, \( V_k \cap V_i = \{0\} \). Therefore, \( g \) is the direct sum (as a vector space) of \( V_k \) and \( V_i \). It is clear, from Proposition 2.3, that both \( V_k \) and \( V_i \) are Lie subalgebras of \( g \). Furthermore, if \( X = \phi(Z) \in V_i \) and \( Y \in V_k \), then, again applying Proposition 2.3, \( [X,Y] = \phi[Z,Y] = [Z,\phi(Y)] = 0 \). Hence, \( g = V_k \oplus V_i \) as a Lie algebra.

Theorem 2.8. Every para-f-Lie group \( G \) is the quotient of the product of an almost product Lie group and a Lie group with trivial para-f-structure by a discrete subgroup.

Proof. Let \( V_k \) and \( V_i \) be subalgebras (defined in Proposition 2.7) of the Lie algebra \( g \) of a para-f-Lie group \( G \). From Proposition 2.7, \( g \) is the Lie algebra direct sum of \( V_k \) and \( V_i \). Using Propositions 1.4 and 1.5, we obtain the theorem from [4].

Remark 2.9. Since a para-f-structure with parallelizable kernel [2] is an almost \( r \)-paracontact structure [1], some examples of almost \( r \)-paracontact structures are used in [3] to illustrate para-f-Lie groups.

References


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