THE LATTICE-ISOMETRIC COPIES OF $\ell_\infty(\Gamma)$ IN QUOTIENTS OF BANACH LATTICES

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Let $E$ be a Banach lattice and let $M$ be a norm-closed and Dedekind $\sigma$-complete ideal of $E$. If $E$ contains a lattice-isometric copy of $\ell_\infty$, then $E/M$ contains such a copy as well, or $M$ contains a lattice copy of $\ell_\infty$. This is one of the consequences of more general results presented in this paper.

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1. Introduction. Let $E$ be a locally solid linear lattice (Riesz space), for example, a Banach lattice, let $M$ be a closed ideal of $E$, and let $\Gamma$ be an infinite set. In [7, Theorem 1] it is proved that if $E$ contains a lattice copy $U$ of $\ell_\infty$ and $M$ is Dedekind $\sigma$-complete, then $E/M$ or $M$ contains such a copy as well. (Here and in what follows the term lattice copy means both lattice and topological copy, and lattice-isometric copy means both lattice and isometric copy.) This is a lattice-topological version of the classical by now theorem of Drewnowski and Roberts which asserts that the noncontainment of $\ell_\infty$ is a three-space property in the class of Banach spaces (see [3, Theorem 3.2.f] and [4]). A natural question to ask is what happens if $E$ is a Banach lattice and $U$ is a lattice-isometric copy of $\ell_\infty$, that is, can we then expect that $E/M$ or $M$ contains a lattice-isometric copy of $\ell_\infty$? A partial positive answer to this question, even for $\ell_\infty(\Gamma)$ instead of $\ell_\infty$, is given in Theorem 1.1 and Corollary 1.2.

For the basic notions and results regarding Banach lattices we refer the reader to the monographs [2, 6]. For the convenience of the reader we recall some definitions. The lattice $E$ is called Dedekind $\alpha$-complete, where $\alpha$ is an infinite cardinal number, if every subset $V$ of $E$ with $\text{card}(V) \leq \alpha$ and bounded from above has a supremum in $E$; if $\alpha = \aleph_0$ then this notion coincides with the notion of Dedekind $\sigma$-completeness, and $E$ is Dedekind complete provided that it is Dedekind $\alpha$-complete for every $\alpha$ (cf. [1]). If $E = (E, \| \cdot \|)$ is a Banach lattice, then $E_a$ denotes the largest ideal in $E$ such that the norm restricted to $E_a$ is order continuous: $E_a = \{ x \in E : |x| \geq x_s \downarrow 0 \text{ implies } \|x_s\| \to 0 \}$; for example, if $E = \ell_\infty(\Gamma)$, then $E_a = c_0(\Gamma)$. We have that $E_a$ is both norm-closed in $E$ and Dedekind complete, and it does not contain any of the lattice copy of $\ell_\infty$ (see, e.g., [6, Proposition 2.4.10 and Corollary 2.4.3]). If $M$ is an ideal of $E,$
then \( Q \) denotes the natural quotient mapping from \( E \) onto the lattice \( E/M \); we have that \( Q \) is a lattice homomorphism, that is, \(|Q(x)| = Q(|x|)\) for all \( x \in E \). 

Our main result reads as follows.

**Theorem 1.1.** Let \( \Gamma \) denote a set with \( \text{card}(\Gamma) \geq \alpha \geq \aleph_0 \). Let \( E \) be a Banach lattice and let \( M \) be a norm-closed and Dedekind \( \alpha \)-complete ideal of \( E \). If \( E \) contains a lattice-isometric copy of \( \ell_\infty(\Gamma) \), then the following alternative holds:

(i) \( E/M \) contains such a copy as well, or
(ii) \( M \) contains a lattice copy of \( \ell_\infty(A) \), where \( \text{card}(A) = \alpha \).

Since Dedekind \( \alpha \)-completeness is inherited by order ideals, the following corollary is an immediate consequence of the theorem.

**Corollary 1.2.** Let \( M \) be a norm-closed ideal of a Dedekind complete (resp., Dedekind \( \sigma \)-complete) Banach lattice \( E \). If \( E \) contains a lattice-isometric copy of \( \ell_\infty(\Gamma) \), then \( E/M \) contains such a copy as well or \( M \) contains a lattice copy of \( \ell_\infty(\Gamma) \) (resp., of \( \ell_\infty \)).

Since the ideal \( M = E_a \) contains no lattice copy of \( \ell_\infty \), from Corollary 1.2 we immediately obtain the following corollary.

**Corollary 1.3.** Let \( E \) be a Banach lattice. If \( E \) contains a lattice-isometric copy of \( \ell_\infty(\Gamma) \), then \( E/E_a \) contains such a copy as well.

In particular, the quotient Banach lattice \( \ell_\infty(\Gamma)/c_0(\Gamma) \) contains a lattice-isometric copy of \( \ell_\infty(\Gamma) \).

Corollaries 1.2 and 1.3 apply for Orlicz spaces endowed with the Luxemburg norm (which form a nontrivial sample class of Dedekind complete Banach lattices, and which contain lattice-isometric copies of \( \ell_\infty \) whenever their norms are not order continuous); one can obtain similar results for Musielak-Orlicz spaces, Lorentz-Orlicz spaces, and Calderón-Lozanovsky spaces (see [5, page 526]).

2. Proof of Theorem 1.1. The symbol \( e_y \) denotes the \( y \)th unit vector of \( \ell_\infty(\Gamma) \), and if \( B \subset \Gamma \) then \( e_B \) denotes the element \( \sup_{y \in B} e_y \). The proof of the theorem depends essentially on the following lemma.

**Lemma 2.1.** Let \( A \) be a set with \( \text{card}(A) = \alpha \geq \aleph_0 \), and let \( M \) be a Dedekind \( \alpha \)-complete and norm-closed ideal of a normed lattice \( E \). If there exist \( u \in E^+ \) and a set \( \{u_a : a \in A\} \) of pairwise disjoint elements of \( E^+ \) such that

(a) \( u_a \leq u \) for all \( a \in A \),
(b) \( b_A := \inf_{a \in A} \|u_a\| > \|Qu\|_{E/M} \),

then \( M \) contains a lattice copy of \( \ell_\infty(A) \).

**Proof.** We partially follow an idea of the proof of [7, Proposition 1(b)]. By (b), there exists \( v \in M \) such that

\[
\|u - v\| < b_A.
\] (2.1)
From the inequality $|u - v| \geq |u| - |v|$ we may assume that $v \geq 0$, and from (a) and (2.1) we obtain that $v \neq 0$. For every $a \in A$, we define $v_a := v \land u_a$ (notice that $v_a \in M$ for all $a \in A$ since $M$ is an ideal). From the equality $u_a = u_a \land u$ for all $a \in A$, the triangle inequality, and from the inequality $|x_1 \land y - x_2 \land y| \leq |x_1 - x_2|$, which holds in every linear lattice (see [2, Theorem 1.6]), we obtain

$$||v_a|| \geq ||u_a|| - ||u - v|| \geq b_A - ||u - v||. \quad (2.2)$$

By (2.1), the number $c := b_A - ||u - v||$ is positive, thus from (2.2) we obtain that $||v_a|| \geq c$ for all $a \in A$. Since the elements $(v_a)_{a \in A}$ are pairwise disjoint and dominated by $v$, and since $M$ is Dedekind $\alpha$-complete, we can define an additive function $R_0$ from the cone $\ell_\infty(A)^+$ into $M$ by the rule $R_0(\xi) := \sup_{a \in A} \xi_a v_a$, where $\xi = (\xi_a)_{a \in A}$, with

$$\xi_a v_a \leq R_0(\xi) \leq ||\xi||_{\ell_\infty(A)} v \quad \forall a \in A. \quad (2.3)$$

By [2, Theorem 1], the formula $R(\xi) := R_0(\xi^+) - R_0(\xi^-)$, $\xi \in \ell_\infty(A)$, defines a linear (positive) mapping from $\ell_\infty(A)$ into $M$. Moreover, since for every $\xi \in \ell_\infty(A)$ the elements $R_0(\xi^+)$ and $R_0(\xi^-)$ are disjoint, we have $|R(\xi)| = R(|\xi|)$, that is, $R$ is a lattice homomorphism; in particular, the range of $R$ is a linear sublattice of $M$ (see [2, page 88]). Finally, from (2.3) we obtain that $c ||\xi||_{\ell_\infty(A)} \leq ||R(\xi)|| \leq ||\xi||_{\ell_\infty(A)} ||v||$ for all $\xi \in \ell_\infty(A)$, and thus (see [2, page 89]) $R$ is a lattice-topological isomorphism.

**Proof of Theorem 1.1.** Let $\Gamma = \bigcup_{\omega \in \Omega} \Gamma_\omega$, where $\text{card}(\Gamma_\omega) = \alpha$ for all $\omega \in \Omega$, $\text{card}(\Omega) = \text{card}(\Gamma)$, and $\Gamma_{\omega_1} \cap \Gamma_{\omega_2} = \emptyset$ for distinct $\omega_1, \omega_2 \in \Omega$. Let $T$ be a lattice-isometric embedding of $\ell_\infty(\Gamma)$ into $E$. We will show that the opposite of (ii) implies (i).

Put $u^\omega := T e_{\omega}$ and $u^\gamma := T e_{\gamma}$, where $\gamma \in \Gamma_\omega$. We have that, for every $\omega \in \Omega$, the element $u^\omega$ and the set $\{u^\omega_y : y \in \Gamma_\omega\}$ fulfill Lemma 2.1(a), and, by hypothesis, the ideal $M$ contains no copy of $\ell_\infty(\Gamma_\omega)$. Thus, from the lemma we obtain that

$$||Qu^\omega|| = 1 \quad \forall \omega \in \Omega. \quad (2.4)$$

Put $W := \{x \in \ell_\infty(\Omega) : x|_{\Gamma_\omega} = \text{const} \}$, and let $H$ denote the linear-lattice isometry from $\ell_\infty(\Omega)$ onto $W$ of the form $H(f_\omega) = e_{\Gamma_\omega}$, $\omega \in \Omega$, where $f_\omega$ is the $\omega$th unit vector of $\ell_\infty(\Omega)$. We claim that the quotient mapping $Q : E \to E/M$ restricted to $T(W)$ is an isometry (and hence, since $Q$ is a lattice homomorphism, it is a lattice isometry). To this end, let $v = \sup_{\omega \in \Omega} \lambda_\omega u^\omega$, where $(\lambda_\omega)_{\omega \in \Omega} \in \ell_\infty(\Omega)^+$ and the sup is taken in the lattice $T(W)$. We obviously have $v = T(\sup_{\omega \in \Omega} \lambda_\omega e_{\Gamma_\omega})$, whence $||Q(v)|| \leq ||v|| = ||\sup_{\omega \in \Omega} \lambda_\omega e_{\Gamma_\omega}|| = \sup_{\omega \in \Omega} \lambda_\omega$. On the other hand, we have that $v \geq \lambda_\omega u^\omega$ for all $\omega \in \Omega$, and by (2.4), we obtain
\[ \|Q(v)\| \geq \sup_{\omega \in \Omega} \lambda_\omega. \]

Finally, \( \|Q(v)\| = \|v\| \) for all positive \( v \), and hence (since \( Q \) is a lattice homomorphism) we obtain \( \|Q(v)\| = \|\|Q(v)\|| = \|Q(|v|)\| = \|\|v\|| = \|v\| \), as claimed. Since \( T, H \), and \( Q|T(W) \) are lattice isometries, the operator \( QT^H \) is a lattice isometry from \( \ell_\infty(\Omega) \) into \( E/M \), and since \( \text{card}(\Omega) = \text{card}(\Gamma) \), the proof is complete.

We want to point out that Lemma 2.1 can also be used to prove the following generalization, to the Banach lattice case, of the main result of [7] quoted in the introduction. Under the same assumptions on \( E, M, \Gamma, \) and \( \alpha \) as in the theorem, let \( T : \ell_\infty(\Gamma) \to E \) be a lattice-topological isomorphism. We define the number

\[ b := \inf \{ \|Q(Te_A)\|_{E/M} : A \subset \Gamma \text{ and } \text{card} A = \alpha \}. \]

Then \( E/M \) contains a lattice copy of \( \ell_\infty(\Gamma) \) (whenever \( b > 0 \); then we mimic the above part of the proof of the theorem with inequality \( \|Qu_\omega\| \geq b \) instead of (2.4)), or \( M \) contains a lattice copy of \( \ell_\infty(A) \) (whenever \( b = 0 \); then we directly apply the lemma).

References


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