The concept of fuzzy multiply positive BCC-ideals of BCC-algebras is introduced, and then some related results are obtained. Moreover, we introduce the concept of \( T \)-fuzzy multiply positive implicative BCC-ideals of BCC-algebras and investigate \( T \)-product of \( T \)-fuzzy multiply positive implicative BCC-ideals of BCC-algebras, examining its properties. Using a \( t \)-norm \( T \), the direct product and \( T \)-product of \( T \)-fuzzy multiply positive implicative BCC-ideals of BCC-algebras are discussed and their properties are investigated.

2000 Mathematics Subject Classification: 06F35, 03G25, 03B52.

1. Introduction and preliminaries. A BCK-algebra is an important class of logical algebras introduced by K. Iséki in 1966. After that, Iséki posed an interesting problem (solved by Wroński [8]) of whether the class of BCK-algebra is a variety. In connection with this problem, Komori [6] introduced a notion of BCC-algebras and Dudek [5] redefined it by using a dual form of the ordinary definition in the sense of Komori. In 1965, Zadeh introduced the notion of fuzzy sets [9]. At present, this concept has been applied to many mathematical branches such as group, functional analysis, probability theory and topology, and so on. In 1991, Ougen applied this concept to BCK-algebras [7], and also many fuzzy structures in BCC-algebras are considered. In this paper, the concept of fuzzy multiply positive implicative BCC-ideals of BCC-algebras is introduced, and some related results are obtained. Moreover, we introduce the concept of \( T \)-fuzzy multiply positive implicative BCC-ideals of BCC-algebras, investigating its properties. Using a \( t \)-norm \( T \), the direct product and \( T \)-product of \( T \)-fuzzy multiply positive implicative BCC-ideals of BCC-algebras are discussed, and their properties are investigated.

By a BCC-algebra, we mean a nonempty set \( G \) with a constant 0 and a binary operation \( \ast \) satisfying the following conditions:

(I) \( ((x \ast y) \ast (z \ast y)) \ast (x \ast z) = 0 \),
(II) \( x \ast x = 0 \),
(III) \( 0 \ast x = 0 \),
(IV) \( x \ast 0 = x \),
(V) \( x \ast y = 0 \) and \( y \ast x = 0 \) imply \( x = y \) for all \( x, y, z \in G \).
On any BCC-algebra, one can define the partial ordering \( \leq \) by putting \( x \leq y \) if and only if \( x \ast y = 0 \).

A BCK-algebra is a BCC-algebra, but there are not BCC-algebra which are not BCK-algebras (cf. [5]). Note that a BCC-algebra \( X \) is a BCK-algebra if and only if it satisfies \( (x \ast y) \ast z = (x \ast z) \ast y \) for all \( x, y, z \in X \).

A nonempty subset \( A \) of a BCC-algebra \( G \) is called a BCC-ideal if (i) 0 \( \in \) \( A \) and (ii) \((x \ast y) \ast z \in A \) and \( y \in A \) imply \( x \ast z \in A \). For any elements \( x \) and \( y \) of a BCC-algebra, \( x \ast y^n \) denotes \( (\cdots ((x \ast y) \ast y) \ast \cdots) \ast y \) in which \( y \) occurs \( n \) times. A nonempty subset \( A \) of a BCC-algebra \( G \) is called an \( n \)-fold BCC-ideal of \( G \) if (i) 0 \( \in \) \( A \) and (ii) for every \( x, y, z \in G \), there exists a natural number \( n \) such that \( x \ast z^n \in A \) whenever \( (x \ast y) \ast z^n \in A \) and \( y \in A \).

We now review some fuzzy logical concepts. A fuzzy set in set \( G \) is a function \( \mu: G \to [0, 1] \). For a fuzzy set \( \mu \) in \( G \) and \( \alpha \in [0, 1] \), define \( \mu_\alpha = \{ x \in G \mid \mu(x) \geq \alpha \} \) which is called a level set of \( G \). A fuzzy set \( \mu \) in a BCC-algebra \( G \) is called a fuzzy BCC-ideal of \( G \) if (i) \( \mu(0) \geq \mu(x) \) and (ii) \( \mu(x \ast y) \geq \min\{\mu((x \ast y) \ast z), \mu(y)\} \) for all \( x, y, z \in G \). A fuzzy set \( \mu \) in a BCC-algebra \( G \) is called an \( n \)-fold fuzzy BCC-ideal of \( G \) if (i) \( \mu(0) \geq \mu(x) \) for all \( x \in G \) and (ii) for every \( x, y, z \in G \), there exists a natural number \( n \) such that \( \mu(x \ast z^n) \geq \min\{\mu((x \ast y) \ast z^n), \mu(y)\} \).

**2. Fuzzy multiply positive implicative BCC-ideals**

**DEFINITION 2.1.** A nonempty subset \( A \) of a BCC-algebra \( G \) is called a multiply positive implicative BCC-ideal of \( G \) if

(i) 0 \( \in \) \( A \),

(ii) for every \( x, y, z \in X \), there exists a natural number \( k = k(x, y, z) \) such that \( x \ast z^k \in A \) whenever \( (x \ast y) \ast z^n \in A \) and \( y \ast z^m \in A \) for any natural numbers \( m \) and \( n \).

**EXAMPLE 2.2.** (i) Consider a BCC-algebra \( G = \{0, 1, 2, 3, 4, 5\} \) with the Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( G \) is a proper BCC-algebra since \((4 \ast 5) \ast 2 \neq (4 \ast 2) \ast 5 \). It is routine to check that \( A = \{0, 1, 2, 3, 4\} \) is a multiply positive implicative BCC-ideal of \( G \).
(ii) Consider a BCC-algebra $G = \{0, a, b, c, d\}$ with the Cayley table as follows:

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
</tbody>
</table>

Then $G$ is a proper BCC-algebra since $(c*a)*d \neq (c*d)*a$. It is routine to check that $A = \{0, a, b, c\}$ is a multiply positive implicative BCC-ideal of $G$.

(iii) Consider a BCC-algebra $G = \{0, a, b, c, 1\}$ with the Cayley table as follows:

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $G$ is a proper BCC-algebra since $(1*b)*a \neq (1*a)*b$. Let $A = \{0, b, c\}$, then $A$ is not a multiply positive implicative BCC-ideal of $G$ because $(1*c)*0^n = c*0^m = c \in A$ while $1*0^k = 1 \notin A$.

**Definition 2.3.** A fuzzy set $\mu$ in a BCC-algebra $G$ is called a fuzzy multiply positive implicative BCC-ideal of $G$ if

(i) $\mu(0) \geq \mu(x)$ for all $x \in G$,

(ii) for any $n, m \in \mathbb{N}$, there exists a natural number $k = k(x, y, z)$ such that $\mu(x*z^k) \geq \min\{\mu((x*y)*z^n), \mu(y*z^m)\}$ for all $x, y, z \in G$.

**Example 2.4.** (i) Consider a BCC-algebra $G = \{0, 1, 2, 3, 4\}$ with the Cayley table as follows:

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

It is a proper BCC-algebra since $(3*1)*2 \neq (3*2)*1$. Define a fuzzy set $\mu$ in $G$ by $\mu(4) = 0.3$ and $\mu(x) = 0.8$ for all $x \neq 4$. Then $\mu$ is a fuzzy multiply positive implicative BCC-ideal of $G$. 


(ii) Let $G$ be a proper BCC-algebra as (i) and let $\mu$ be a fuzzy set in $G$ defined by

$$
\mu(x) = \begin{cases} 
\alpha_1 & \text{if } x \in \{0, 2, 3\}, \\
\alpha_2 & \text{otherwise}, 
\end{cases}
$$

where $\alpha_1 > \alpha_2$ in $[0, 1]$. It is easy to check that $\mu$ is not a fuzzy multiply positive implicative BCC-ideal of $G$ because $\mu(4 \ast 0^k) = \mu(4) = \alpha_2 \leq \min\{\mu((4 \ast 3) \ast 0^n), \mu(3 \ast 0^m)\}$ for any positive integer numbers $m, n,$ and $k$.

**Theorem 2.5.** Let $\mu$ be a fuzzy set in a BCC-algebra $G$, then $\mu$ is a fuzzy multiply positive implicative BCC-ideal of $G$ if and only if the nonempty level set $\mu(\alpha) = \{x \in G \mid \mu(x) \geq \alpha\}$ of $\mu$ is a multiply positive implicative BCC-ideal of $G$.

**Proof.** Suppose that $\mu$ is a fuzzy multiply positive implicative BCC-ideal of $G$ and $\mu(\alpha) \neq \emptyset$ for any $\alpha \in [0, 1]$. Then there exists $x \in \mu(\alpha)$ and so $\mu(x) \geq \alpha$. It follows that $\mu(0) \geq \mu(x) \geq \alpha$ so that $0 \in \mu(\alpha)$. Let $x, y, z \in G$ be such that $(x \ast y) \ast z^n \in \mu(\alpha)$ and $y \ast z^m \in \mu(\alpha)$. By **Definition 2.3**, there exists a natural number $k$ such that $\mu((x \ast z)^k) \geq \min\{\mu((x \ast y) \ast z^n), \mu(y \ast z^m)\} \geq \min\{\alpha, \alpha\} = \alpha$ and that $x \ast z^k \in \mu(\alpha)$. Hence $\mu(\alpha)$ is a multiply positive implicative BCC-ideal of $G$. Conversely, assume that $\mu(\alpha)$ is a multiply positive implicative BCC-ideal of $G$ for every $\alpha \in [0, 1]$. For any $x \in G$, let $\mu(x) = \alpha$. Then $x \in \mu(\alpha)$. Since $0 \in \mu(\alpha)$, it follows that $\mu(0) \geq \alpha = \mu(x)$ so that $\mu(0) \geq \mu(x)$ for all $x \in G$. Now suppose that there exist $x_0, y_0, z_0 \in G$ such that $\mu(x_0 \ast z_0^k) < \min\{\mu((x_0 \ast y_0) \ast z_0), \mu(y_0 \ast z_0^m)\}$. Let $\lambda_0 = \mu(x_0 \ast z_0^k) + \min\{\mu((x_0 \ast y_0) \ast z_0), \mu(y_0 \ast z_0^m)\} / 2$, then $\lambda_0 > \mu(x_0 \ast z_0^k)$ and $0 \leq \lambda_0 < \min\{\mu((x_0 \ast y_0) \ast z_0^k), \mu(y_0 \ast z_0^m)\} \leq 1$, so we have $\mu((x_0 \ast y_0) \ast z_0^k) \geq \lambda_0$ and $\mu(y_0 \ast z_0^m) \geq \lambda_0$, then $(x_0 \ast y_0) \ast z_0^k \in \mu(\lambda_0)$ and $y_0 \ast z_0^m \in \mu(\lambda_0)$. As $\mu(\lambda_0)$ is a multiply positive BCC-ideal of $G$, it implies $x_0 \ast z_0^k \in \mu(\lambda_0)$ and $x_0 \ast z_0^k \geq \lambda_0$. This is a contradiction. Hence $\mu$ is a fuzzy multiply positive implicative BCC-ideal of $G$. \hfill $\Box$

**Theorem 2.6.** Let $A$ be a nonempty subset of a BCC-algebra $G$, and $\mu$ a fuzzy set in $G$ defined by

$$
\mu(x) = \begin{cases} 
\alpha_1 & \text{if } x \in A, \\
\alpha_2 & \text{otherwise}, 
\end{cases}
$$

where $\alpha_1 > \alpha_2$ in $[0, 1]$. Then $\mu$ is a fuzzy multiply positive implicative BCC-ideal of $G$ if and only if $A$ is a multiply positive implicative BCC-ideal of $G$.

**Proof.** Assume that $\mu$ is a fuzzy multiply positive implicative BCC-ideal of $G$. Since $\mu(0) \geq \mu(x)$ for all $x \in G$, we have $\mu(0) = \alpha_1$ and so $0 \in A$. Let $x, y, z \in G$ be such that $(x \ast y) \ast z^n \in A$ and $y \ast z^m \in A$. By **Definition 2.3**, there exists a natural number $k = k(x, y, z)$ such that $\mu((x \ast y) \ast z^n) \geq \min\{\mu(x \ast z^k) \geq \min\{\mu((x \ast y) \ast z^n), \mu(y \ast z^m)\} = \alpha_1$ and that $x \ast z^k \in A$. Hence $A$ is a multiply positive implicative BCC-ideal of $G$.
Conversely, suppose that $A$ is a multiply positive implicative BCC-ideal of $G$. Since $0 \in A$, it follows that $\mu(0) = \alpha_1 \geq \mu(x)$ for all $x \in G$. Let $x, y, z \in G$. If $y * z^m \notin A$ and $(x * y) * z^n \in A$, then clearly $\mu(x * y) * z^n \geq \min\{\mu((x * y) * z^n), \mu(y * z^m)\}$. Assume that $y * z^m \in A$ and $(x * y) * z^n \notin A$, we have $(x * y) * z^k \notin A$. Therefore $\mu(x * z^k) = \alpha_2 = \min\{\mu((x * y) * z^n), \mu(y * z^m)\}$.

Hence, $\mu$ is a fuzzy multiply positive implicative BCC-ideal of $G$.

A fuzzy relation on any set $S$ is a fuzzy subset $\mu : S \times S \to [0, 1]$. If $\mu$ is a fuzzy relation on a set $S$ and $\nu$ is a fuzzy subset of $S$, then $\mu$ is a fuzzy relation on $\nu$ if $\mu(x, y) \leq \min\{\nu(x), \nu(y)\}$ for all $x, y \in S$. Let $\mu$ and $\nu$ be defined as $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$. One can prove that $\mu \times \nu$ is a fuzzy relation on $S$ and $(\mu \times \nu)_t = \mu_t \times \nu_t$ for all $t \in [0, 1]$. If $\mu$ is a fuzzy subset of a set $S$, the strongest fuzzy relation on $S$ that is a fuzzy relation on $\nu$ is $\mu_\nu$, given by $\mu_\nu(x, y) = \min\{\mu(x), \nu(y)\}$ for all $x, y \in S$. In this case we have $(\mu_\nu)_t = \nu_t \times \nu_t$ for all $t \in [0, 1]$ (see [2]).

**Theorem 2.7.** For a given fuzzy subset $\nu$ of a BCC-algebra $G$, let $\mu_\nu$ be the strongest fuzzy relation on $G$. If $\mu_\nu$ is a fuzzy multiply positive implicative BCC-ideal of $G \times G$, then $\nu(0) \geq \nu(x)$ for all $x \in G$.

**Proof.** Since $\mu_\nu$ is a fuzzy multiply positive implicative BCC-ideal of $G \times G$, it follows that $\mu_\nu(0, 0) \geq \mu_\nu(x, x)$ for all $x \in G$. This means that $\min\{\nu(0), \nu(0)\} \geq \min\{\nu(x), \nu(x)\}$, which implies that $\nu(0) \geq \nu(x)$.

**Theorem 2.8.** If $\nu$ is a fuzzy multiply positive implicative BCC-ideal of a BCC-algebra $G$, then the level multiply positive implicative BCC-ideals $(\mu_\nu)_t$ are given by

$$(\mu_\nu)_t = \mu_t \times \nu_t \quad \forall t \in [0, 1].$$

The proof is obvious.

**Theorem 2.9.** If $\mu$ and $\nu$ are fuzzy multiply positive implicative BCC-ideals of a BCC-algebra $G$, then $\mu \times \nu$ is a fuzzy multiply positive implicative BCC-ideal of $G \times G$.

**Proof.** For any $(x, y) \in G \times G$,

$$(\mu \times \nu)(0, 0) = \min\{\mu(0), \nu(0)\} \geq \min\{\mu(x), \nu(x)\} = (\mu \times \nu)(x, y).$$

Now, let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2) \in G \times G$. For any $n, m \in \mathbb{N}$, there exists a natural number $k$ such that

$$(\mu \times \nu)(x * z^k) = (\mu \times \nu)((x_1, x_2) * (z_1, z_2)^k) = (\mu \times \nu)(x_1 * z_1^{k}, x_2 * z_2^{k}) \geq \min\{\mu(x_1 * z_1^{k}), \nu(x_2 * z_2^{k})\}$$
\[
(\mu(x)\times \nu(x)) = \min \{ \mu((x_1 \ast y_1) \ast z_1^n), \mu(y_1 \ast z_1^m) \}, \\
\min \{ \nu((x_1 \ast y_2) \ast z_2^n), \nu(y_2 \ast z_2^m) \}
\]
\[
= \min \{ \mu((x_1 \ast y_1) \ast z_1^n), \nu((x_2 \ast y_2) \ast z_2^n) \}, \\
\min \{ \mu(y_1 \ast z_1^m), \nu(y_2 \ast z_2^m) \}
\]
\[
= \min \left\{ (\mu \times \nu)((x_1 \ast y_1) \ast (y_1 \ast y_2) \ast (z_1 \ast z_2)^n) \right\}, \\
(\mu \times \nu)((y_1 \ast y_2) \ast (z_1 \ast z_2)^m)
\]
\[
= \min \left\{ (\mu \times \nu)((x_1 \ast y_1) \ast (y_1 \ast y_2) \ast (z_1 \ast z_2)^n), (\mu \times \nu)((x_2 \ast y_2) \ast (y_1 \ast y_2) \ast (z_2 \ast z_2)^n), \\
(\mu \times \nu)((y_1 \ast z_1^m) \ast (y_2 \ast z_2^m)) \right\}.
\]

(2.5)

Hence \( \mu \times \nu \) is a fuzzy multiply positive implicative BCC-ideals of \( G \times G \). ☐

**Theorem 2.10.** Let \( \mu \) and \( \nu \) be fuzzy subsets of a BCC-algebra \( G \) such that \( \mu \times \nu \) is a fuzzy multiply positive implicative BCC-ideal of \( G \times G \). Then

(i) either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \nu(0) \) for all \( x \in G \),

(ii) if \( \mu(x) \leq \mu(0) \) for all \( x \in G \), then either \( \mu(x) \leq \nu(0) \) or \( \nu(x) \leq \nu(0) \),

(iii) if \( \nu(x) \leq \nu(0) \) for all \( x \in G \), then either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \mu(0) \),

(iv) either \( \mu \) or \( \nu \) is a fuzzy multiply positive implicative BCC-ideal of \( G \).

**Proof.**

(i) Suppose that \( \mu(x) > \mu(0) \) and \( \nu(x) > \nu(0) \) for some \( x, y \in G \). Then \( (\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \min\{\mu(0), \nu(0)\} = (\mu \times \nu)(0, 0) \). This is a contradiction and we obtain (i).

(ii) Assume that there exist \( x, y \in G \) such that \( \mu(x) > \nu(0) \) and \( \nu(y) > \nu(0) \). Then \( (\mu \times \nu)(0, 0) = \min\{\mu(0), \nu(0)\} = \nu(0) \). It follows that \( (\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \nu(0) = (\mu \times \nu)(0, 0) \). This is a contradiction. Hence (ii) holds.

(iii) Item (iii) is proved by similar method to part (ii).

(iv) Since by (i), either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \nu(0) \) for all \( x \in G \), without loss of generality, we may assume that \( \nu(x) \leq \nu(0) \) for all \( x \in G \). Form (iii), it follows that either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \mu(0) \). If \( \nu(x) \leq \mu(0) \) for all \( x \in G \), then \( (\mu \times \nu)(0, x) = \min\{\mu(0), \nu(x)\} = \nu(x) \). Let \( (x_1, x_2), (y_1, y_2), (z_1, z_2) \in G \times G \). Since \( \mu \times \nu \) is a fuzzy multiply positive implicative BCC-ideal of \( G \times G \), then for any \( n, m \in \mathbb{N} \), there exists a natural number \( k \) such that

\[
(\mu \times \nu)((x_1, x_2) \ast (z_1, z_2)^k)
\]
\[
\geq \min \left\{ (\mu \times \nu)(((x_1, x_2) \ast (y_1, y_2)) \ast (z_1, z_2)^n), \\
(\mu \times \nu)((y_1, y_2) \ast (z_1, z_2)^m) \right\}
\]
\[
= \min \left\{ (\mu \times \nu)(((x_1 \ast y_1) \ast z_1^n), ((x_2 \ast y_2) \ast z_2^n)), \\
(\mu \times \nu)((y_1 \ast z_1^m) \ast (y_2 \ast z_2^m)) \right\}.
\]
If we take $x_1 = y_1 = z_1 = 0$, then
\[
\nu(x_2 \ast z_2^k) = (\mu \times \nu)(0, x_2 \ast z_2^k)
\]
\[
= (\mu \times \nu)((0, x_2) \ast (0, z_2)^k)
\]
\[
\geq \min \{ (\mu \times \nu)((0, x_2 \ast y_2) \ast z_2^n), (\mu \times \nu)(0, y_2 \ast z_2^m) \}
\]
\[
= \min \{ \min \{ \mu(0), \nu((x_2 \ast y_2) \ast z_2^n) \}, \min \{ \nu(0), \nu(y_2 \ast z_2^m) \} \}
\]
\[
= \min \{ \nu((x_2 \ast y_2) \ast z_2^n), \nu(y_2 \ast z_2^m) \}.
\]
(2.7)

This proves that $\nu$ is a fuzzy multiply positive BCC-ideal of $G$. Now we consider the case $\mu(x) \leq \mu(0)$ for all $x \in G$. Suppose that $\nu(y) > \mu(0)$ for some $y \in G$. Then $\nu(0) \geq \nu(y) > \mu(0)$. Since $\mu(0) \geq \mu(x)$ for all $x \in G$, it follows that $\nu(0) > \mu(x)$ for any $x \in G$. Hence $(\mu \times \nu)(x, 0) = \min\{\mu(x), \nu(0)\} = \mu(x)$. Taking $x_2 = y_2 = z_2 = 0$ in (2.6), then
\[
\mu(x_1 \ast z_1^k) = (\mu \times \nu)(x_1 \ast z_1^k, 0)
\]
\[
= (\mu \times \nu)((x_1, 0) \ast (z_1, 0)^k)
\]
\[
\geq \min \{ (\mu \times \nu)((x_1 \ast y_1) \ast z_1^n), (\mu \times \nu)(y_1 \ast z_1^m, 0) \}
\]
\[
= \min \{ \min \{ \mu((x_1 \ast y_1) \ast z_1^n), \nu(0) \}, \min \{ \mu(y_1 \ast z_1^m), \nu(0) \} \}
\]
\[
= \min \{ \mu((x_1 \ast y_1) \ast z_1^n), \mu(y_1 \ast z_1^m) \}
\]
(2.8)

which proves that $\mu$ is a fuzzy multiply positive implicative BCC-ideal of $G$. □

**Theorem 2.11.** Let $\nu$ be a fuzzy subset of a BCC-algebra $G$ and let $\mu_\nu$ be the strongest fuzzy relation on $G$. Then $\nu$ is a fuzzy multiply positive implicative BCC-ideal of $G$ if and only if $\mu_\nu$ is a fuzzy multiply positive implicative BCC-ideal of $G \times G$.

**Proof.** Assume that $\nu$ is a fuzzy multiply positive implicative BCC-ideal of $X$, then
\[
\mu_\nu(0, 0) = \min \{ \nu(0), \nu(0) \} \geq \min \{ \nu(x), \nu(y) \} = \mu_\nu(x, y)
\]
(2.9)

for any $(x, y) \in G \times G$. Moreover, for any $n, m \in \mathbb{N}$, there exists a natural number $k$ such that
\[
\mu_\nu((x_1, x_2) \ast (z_1, z_2)^k) = \mu_\nu(x_1 \ast z_1^k, x_2 \ast z_2^k)
\]
\[
= \min \{ \nu(x_1 \ast z_1^k), \nu(x_2 \ast z_2^k) \}
\]
\[
\geq \min \{ \min \{ \nu((x_1 \ast y_1) \ast z_1^n), \nu(y_1 \ast z_1^m) \}, \min \{ \nu((x_2 \ast y_2) \ast z_2^n), \nu(y_2 \ast z_2^m) \} \}
\]
\[
\begin{align*}
&= \min \{ \min \{ \nu((x_1 \ast y_1) \ast z_1^n), \nu((x_2 \ast y_2) \ast z_2^n) \}, \\
&\quad \min \{ \nu(y_1 \ast z_1^m), \nu(y_2 \ast z_2^m) \} \}
\end{align*}
\]

\[
= \min \{ \mu_\nu((x_1 \ast y_1) \ast z_1^n), (x_2 \ast y_2) \ast z_2^n), \\
\mu_\nu(y_1 \ast z_1^m, y_2 \ast z_2^m) \}
\]

\[
= \min \{ \mu_\nu((x_1, x_2) \ast (y_1, y_2)) \ast (z_1, z_2)^n), \\
\mu_\nu((y_1, y_2) \ast (z_1, z_2)^m) \}
\]

\[(2.10)\]

for any \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in G \times G.\)

Hence \(\mu_\nu\) is a fuzzy multiply positive implicative BCC-ideal of \(G \times G.\)

Conversely, suppose that \(\mu_\nu\) is a fuzzy multiply positive implicative BCC-ideal of \(G \times G.\) Then for all \((x_1, x_2) \in G \times G,\)

\[
\min \{ \nu(0), \nu(0) \} = \mu_\nu(0, 0) \geq \mu_\nu(x_1, x_2) = \min \{ \nu(x_1), \nu(x_2) \}. \quad (2.11)
\]

It follows that \(\nu(0) \geq \nu(x)\) for all \(x \in G.\) Now, for any \(n, m \in \mathbb{N},\) there exists a natural number \(k\) such that

\[
\begin{align*}
&\min \{ \nu(x_1 \ast z_1^k), \nu(x_2 \ast z_2^k) \} \\
&= \mu_\nu(x_1 \ast z_1^k, x_2 \ast z_2^k) = \mu_\nu((x_1, x_2) \ast (z_1, z_2)^k) \\
&\geq \min \{ \mu_\nu((x_1, x_2) \ast (y_1, y_2)) \ast (z_1, z_2)^n), \mu_\nu((y_1, y_2) \ast (z_1, z_2)^m) \}
\end{align*}
\]

\[
= \min \{ \mu_\nu((x_1 \ast y_1) \ast z_1^n, (x_2 \ast y_2) \ast z_2^n), \mu_\nu(y_1 \ast z_1^m, y_2 \ast z_2^m) \}
\]

\[
= \min \{ \min \{ \nu((x_1 \ast y_1) \ast z_1^n), \nu((x_2 \ast y_2) \ast z_2^n) \}, \\
\min \{ \nu(y_1 \ast z_1^m), \nu(y_2 \ast z_2^m) \} \}
\]

\[
= \min \{ \min \{ \nu((x_1 \ast y_1) \ast z_1^n), \nu(y_1 \ast z_1^m) \}, \\
\min \{ \nu((x_2 \ast y_2) \ast z_2^n), \nu(y_2 \ast z_2^m) \} \}.
\]

\[(2.12)\]

If we take \(x_2 = y_2 = z_2 = 0\) (resp., \(x_1 = y_1 = z_1 = 0\)), then \(\nu(x_1 \ast z_1^k) \geq \min \{ \nu((x_1 \ast y_1) \ast z_1^n), \nu(y_2 \ast z_2^m) \}.\) Hence \(\nu\) is a fuzzy multiply positive implicative BCC-ideal of \(G.\)

\[
\square
\]

3. \(T\)-fuzzy multiply positive implicative BCC-ideals

**Definition 3.1** [1]. By a \(t\)-norm \(T: [0,1] \times [0,1] \rightarrow [0,1]\) satisfying the following conditions:

(I) \(T(x, 1) = x,\)

(II) \(T(x, y) \leq T(x, z)\) if \(y \leq z,\)

(III) \(T(x, y) = T(y, x),\)

(IV) \(T(x, T(y, z)) = T(T(x, y), z)\) for all \(x, y, z \in [0,1].\)

Every \(t\)-norm \(T\) has a useful property \(T(\alpha, \beta) \leq \min \{ \alpha, \beta \}\) for all \(\alpha, \beta \in [0,1].\)
**Lemma 3.2** [1]. Let \( T \) be a \( t \)-norm. Then \( T(T(\alpha,\beta),T(\nu,\delta)) = T(T(\alpha,\nu),T(\beta,\delta)) \) for all \( \alpha,\beta,\nu,\delta \in [0,1] \).

**Definition 3.3.** A fuzzy subset \( \mu : G \rightarrow [0,1] \) in a BCC-algebra \( G \) is called a fuzzy multiply positive implicative BCC-ideal of \( G \) with respect to a \( t \)-norm \( T \) (briefly, \( T \)-fuzzy multiply positive implicative BCC-ideal of \( G \)) if  
(i) \( \mu(0) \geq \mu(x) \) for all \( x \in G \),  
(ii) for any \( n,m \in \mathbb{N} \), there exists a natural number \( k = k(x,y,z) \) such that \( \mu(x \ast z^k) \geq T(\mu((x \ast y) \ast z^n),\mu(y \ast z^m)) \) for any \( x,y,z \in G \).

**Example 3.4.** Consider a BCC-algebra \( G = \{0,1,2,3,4\} \) with the Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

By routine calculation, \( G \) is a proper BCC-algebra (cf. [5]). Define a fuzzy set \( \mu \) by \( \mu(0) = \mu(1) = \mu(2) = \mu(3) = 0.8 \) and \( \mu(4) = 0.3 \). Let \( T(\alpha,\beta) = \max\{\alpha + \beta - 1,0\} \) for all \( \alpha,\beta \in [0,1] \). Then \( T \) is a \( t \)-norm. It is easy to check that \( \mu \) is a \( T \)-fuzzy multiply positive implicative BCC-ideal of \( G \).

**Theorem 3.5.** Let \( \mu \) be a \( T \)-fuzzy multiply positive implicative BCC-ideal of a BCC-algebra \( G \) and let \( \alpha \in [0,1] \) if \( \alpha = 1 \), then the nonempty subset \( \mu_\alpha \) is a multiply positive implicative BCC-ideal of \( G \).

**Proof.** Assume that \( \alpha = 1 \) and \( x \in \mu_\alpha \), then \( \mu(x) \geq 1 \). Thus \( \mu(0) \geq \mu(x) \geq 1 \) and \( 0 \in \mu_\alpha \).

Moreover, suppose that \( (x \ast y) \ast z^n \in \mu_\alpha \) and \( y \ast z^m \in \mu_\alpha \), then \( \mu((x \ast y) \ast z^n) \geq 1 \) and \( \mu(y \ast z^m) \geq 1 \). By **Definition 3.3**, there exists a natural number \( k \) such that \( \mu(x \ast z^k) \geq T(\mu((x \ast y) \ast z^n),\mu(y \ast z^m)) \geq T(1,1) = 1 \) and that \( x \ast z^k \in \mu_\alpha \). Hence \( \mu_\alpha \) is a multiply positive implicative BCC-ideal of \( G \).

For a fuzzy set \( \mu \) on a BCC-algebra \( G \) and a map \( \theta : G \rightarrow G \), we define a mapping \( \mu[\theta] : G \rightarrow [0,1] \) by \( \mu[\theta](x) = \mu(\theta(x)) \) for all \( x \in G \).

**Theorem 3.6.** If \( \mu \) is a \( T \)-fuzzy multiply positive implicative BCC-ideal of a BCC-algebra \( G \) and \( \theta \) is an epimorphism of \( G \), then \( \mu[\theta] \) is a \( T \)-fuzzy multiply positive implicative BCC-ideal of \( G \).
\textbf{Proof.} Let $\mu[\theta](0) = \mu(\theta(0)) = \mu(0) \geq \mu(y)$ for any $y \in G$. Since $\theta$ is an epimorphism of $G$, then there exists $x \in G$ such that $\theta(x) = y$. Thus $\mu[\theta](0) \geq \mu(\theta(x)) = \mu[\theta](x)$. As $y$ is an arbitrary element of $G$, the above result is true for any $x \in G$.

Moreover, for any $n, m \in \mathbb{N}$, there exists a natural number $k$ such that

$$
\mu[\theta](x \ast z^k) = \mu(\theta(x) \ast \theta(z)^k) \\
\geq T(\mu((\theta(x) \ast \theta(y))^\ast \theta(z)^n), \mu(\theta(y) \ast \theta(z)^m)) \\
= T(\mu(\theta((x \ast y) \ast z^n)), \mu(\theta(y \ast z^m))) \\
= T(\mu[\theta]((x \ast y) \ast z^n), \mu[\theta](y \ast z^m)).
$$

(3.1)

Hence $\mu[\theta]$ is a $T$-fuzzy multiply positive implicative BCC-ideal of $G$. \hfill \Box

Let $f$ be a mapping defined on a BCC-algebra $G$. If $\nu$ is a fuzzy set in $f(G)$, then the fuzzy set $\mu_\nu$ of $G$ defined by $\mu(x) = \nu(f(x))$ is called the preimage of $\nu$ under $f$.

\textbf{Theorem 3.7.} An onto homomorphic preimage of a $T$-fuzzy multiply positive implicative BCC-ideal is a $T$-fuzzy multiply positive implicative BCC-ideal.

\textbf{Proof.} Let $f : G \rightarrow G'$ be an onto homomorphism of BCC-algebra, $\nu$ a $T$-fuzzy multiply positive implicative BCC-ideal of $G'$, and $\mu$ the preimage of $\nu$ under $f$. Then $\mu(0) = \nu(f(0)) = \nu(0') \geq \nu(f(x)) = \mu(x)$ for all $x \in G$. Moreover, for any $n, m \in \mathbb{N}$, there exists a natural number $k$ such that

$$
\mu(x \ast z^k) = \nu(f(x) \ast z^k) = \nu(f(x) \ast f(z)^k) \\
\geq T(\nu((f(x) \ast f(y)) \ast f(z)^n), \nu(f(y) \ast f(z)^m)) \\
= T(\nu(f((x \ast y) \ast z^n)), \nu(f(y \ast z^m))) \\
= T(\mu((x \ast y) \ast z^n), \mu(y \ast z^m))
$$

(3.2)

for any $x, y, z \in G$. Hence $\mu$ is a $T$-fuzzy multiply positive implicative BCC-ideal of $G$. \hfill \Box

If $\mu$ is a fuzzy set in a BCC-algebra $G$ and $f$ is a mapping defined on $G$, then the fuzzy set $\mu^f$ in $f(G)$ defined by $\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x)$ for all $y \in G$ is called the image of $\mu$ under $f$. A fuzzy set $\mu$ in $G$ is said to have sup property if, for every subset $T \subseteq G$, there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t \in T} \mu(t)$.

\textbf{Theorem 3.8.} An onto homomorphic image of a $T$-fuzzy multiply positive implicative BCC-ideal with sup property is a $T$-fuzzy multiply positive implicative BCC-ideal.

\textbf{Proof.} Let $f : G \rightarrow G'$ be an onto homomorphism of BCC-algebras and let $\mu$ be a $T$-fuzzy multiply positive implicative BCC-ideal of $G$ with sup property. Then $\mu^f(0) = \sup_{f \in f^{-1}(0)} \mu(t) = \mu(0) \geq \mu(x)$ for any $x \in G$. Furthermore, we
have \( \mu^f(x_1) = \sup_{t \in f^{-1}(x_1)} \mu(t) \) for any \( x_1 \in G' \). Thus \( \mu^f(0) \geq \sup_{t \in f^{-1}(x_1)} \mu(t) = \mu^f(x_1) \) for any \( x_1 \in G' \). Moreover, for any \( x_1, y_1, z_1 \in G' \), let \( x \in f^{-1}(x_1) \), \( y \in f^{-1}(y_1) \), and \( z \in f^{-1}(z_1) \) such that

\[
\begin{align*}
\mu(x \ast z^k) &= \sup_{t \in f^{-1}(x_1 \ast z_1^k)} \mu(t), \\
\mu((x \ast y) \ast z^n) &= \sup_{t \in f^{-1}((x \ast y) \ast z_1^n)} \mu(t), \\
\mu(y \ast z^n) &= \sup_{t \in f^{-1}(y \ast z_1^n)} \mu(t).
\end{align*}
\]

Thus

\[
\begin{align*}
\mu^f(x_1 \ast z_1^k) &= \sup_{t \in f^{-1}(x_1 \ast z_1^k)} \mu(t) = \mu(x \ast z^k) \\
&\geq T(\mu((x \ast y) \ast z^n), \mu(y \ast z^n)) \\
&= T\left(\sup_{t \in f^{-1}((x_1 \ast y_1) \ast z_1^n)} \mu(t), \sup_{t \in f^{-1}(y_1 \ast z_1^n)} \mu(t)\right) \\
&= T(\mu^f((x_1 \ast y_1) \ast z_1^n), \mu^f(y_1 \ast z_1^n)).
\end{align*}
\]

Therefore, \( \mu^f \) is a \( T \)-fuzzy multiply positive implicative BCC-ideal of \( G' \). \( \square \)

4. Fuzzy multiply positive implicative BCC-ideals induced by norms

**Theorem 4.1.** Let \( T \) be a \( t \)-norm and \( G = G_1 \times G_2 \) the direct product BCC-algebra of BCC-algebras \( G_1 \) and \( G_2 \). If \( \mu_1 \) (resp., \( \mu_2 \)) is a \( T \)-fuzzy multiply positive implicative BCC-ideal of \( G_1 \) (resp., \( G_2 \)), then \( \mu = \mu_1 \times \mu_2 \) is a \( T \)-fuzzy multiply positive implicative BCC-ideal of \( G \) defined by \( \mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)) \) for all \( (x_1, x_2) \in G_1 \times G_2 \).

The proof is identical with the corresponding proof from [3].

We will generalize the idea to the product of \( n \) \( T \)-fuzzy multiply positive implicative BCC-ideals. We first need to generalize the domain of \( t \)-norm \( T \) to \( \Pi_{i=1}^n [0, 1] \) as follows.

The function \( T_n : \Pi_{i=1}^n [0, 1] \to [0, 1] \) is defined by

\[
T_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n))
\]

for all \( 1 \leq i \leq n \), where \( n \geq 2 \), \( T_2 = T \), and \( T_1 = \text{id} \) (identity). For a \( t \)-norm \( T \) and every \( \alpha_i, \beta_i \in [0, 1] \), where \( 1 \leq i \leq n \) and \( n \geq 2 \), we have

\[
\begin{align*}
T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \ldots, T(\alpha_n, \beta_n)) &= T(T_n(\alpha_1, \alpha_2, \ldots, \alpha_n), T_n(\beta_1, \beta_2, \ldots, \beta_n)).
\end{align*}
\]
**Theorem 4.2.** Let $T$ be a $t$-norm, $\{G_i\}_{i=1}^n$ the finite collection of BCC-algebras, and $G = \prod_{i=1}^n G_i$ the direct product BCC-algebra of $\{G_i\}$. Let $\mu_i$ be a $T$-fuzzy multiply positive implicative BCC-ideal of $\{G_i\}$, where $1 \leq i \leq n$. Then $\mu = \prod_{i=1}^n \mu_i$ defined by $\mu(x_1, x_2, \ldots, x_n) = (\prod_{i=1}^n \mu_i(x_1, x_2, \ldots, x_n)) = T_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n))$ is a $T$-fuzzy multiply positive implicative BCC-ideal of $G$.

The proof is identical with the corresponding proof from [3].

**Definition 4.3** [4]. Let $T$ be a $t$-norm and let $\mu$ and $\nu$ be fuzzy sets in a BCC-algebra $G$. Then the $T$-product of $\mu$ and $\nu$, written as $[\mu \cdot \nu]_T$, is defined by $[\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x))$ for all $x \in G$.

**Theorem 4.4.** Let $T$ be a $t$-norm and let $\mu$ and $\nu$ be $T$-fuzzy multiply positive implicative BCC-ideals of a BCC-algebra $G$. If $T^*$ is a $t$-norm which dominates $T$, that is, $T^*(T(\alpha, \beta), T(\nu, \delta)) \geq T(T^*(\nu, \delta), T^*(\beta, \delta))$ for all $\alpha, \beta, \nu, \delta \in [0, 1]$, then the $T^*$-product of $\mu$ and $\nu$, $[\mu \cdot \nu]_{T^*}$, is a $T$-fuzzy multiply positive implicative BCC-ideal of $G$.

**Proof.** Let $[\mu \cdot \nu]_{T^*}((0, 0)) = T^*(\mu(0), \nu(0)) \geq T^*(\mu(x), \nu(x)) = [\mu \cdot \nu]_{T^*}(x)$ for any $x \in G$. Moreover, for any $n, m \in \mathbb{N}$, there exists a natural number $k$, such that

$$
[\mu \cdot \nu]_{T^*}(x \cdot z^k) = T^*(\mu(x \cdot z^k), \nu(x \cdot z^k)) \\
\quad \geq T^*(T(\mu((x \cdot y) \cdot z^n), \nu((y \cdot z^m))), T(\nu((x \cdot y) \cdot z^n), \nu((y \cdot z^m)))) \\
\quad \geq T^*(T(\mu((x \cdot y) \cdot z^n), \nu((x \cdot y) \cdot z^n))), T^*(\mu(y \cdot z^m), \nu(y \cdot z^m))) \\
= T([\mu \cdot \nu]_{T^*}((x \cdot y) \cdot z^n), [\mu \cdot \nu]_{T^*}(y \cdot z^m)).
$$

Hence $[\mu \cdot \nu]_{T^*}$ is a $T$-fuzzy multiply positive implicative BCC-ideal of $G$. 

Let $f : G \to G'$ be an onto homomorphism of BCC-algebras. Let $T$ and $T^*$ be $t$-norms such that $T^*$ dominates $T$. If $\mu$ and $\nu$ are $T$-fuzzy multiply positive implicative BCC-ideals of $G'$, then the $T^*$-product of $\mu$ and $\nu$, $[\mu \cdot \nu]_{T^*}$, is a $T$-fuzzy multiply positive implicative BCC-ideal of $G'$. Since every onto homomorphism preimage of a $T$-fuzzy multiply positive implicative BCC-ideal is a $T$-fuzzy multiply positive implicative BCC-ideal, the preimages $f^{-1}(\mu)$, $f^{-1}(\nu)$, and $f^{-1}([\mu \cdot \nu]_{T^*})$ are $T$-fuzzy multiply positive implicative BCC-ideals of $G$. The next theorem provides the relation between $f^{-1}([\mu \cdot \nu]_{T^*})$ and $T^*$-product $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

**Theorem 4.5.** Let $f : G \to G'$ be an onto homomorphism of BCC-algebras. Let $T$ and $T^*$ be $t$-norms such that $T^*$ dominates $T$. Let $\mu$ and $\nu$ be $T$-fuzzy multiply positive implicative BCC-ideals of $G'$. If $[\mu \cdot \nu]_{T^*}$ is the $T^*$-product of $\mu$ and $\nu$, and $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ is the $T^*$-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then $f^{-1}([\mu \cdot \nu]_{T^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$. 
ACKNOWLEDGMENT. The authors would like to thank the referees for their valuable suggestions.

REFERENCES


Jianming Zhan: Department of Mathematics, Hubei Institute for Nationalities, Enshi, Hubei Province 445000, China
E-mail address: zhanjianming@hotmail.com

Zhisong Tan: Department of Mathematics, Hubei Institute for Nationalities, Enshi, Hubei Province 445000, China
E-mail address: es-tzs@263.net