W. Freedman introduced an alternate to the Dunford-Pettis property, called the DP1 property, in 1997. He showed that for $1 \leq p < \infty$, $(\bigoplus_{\alpha \in \mathcal{A}} X_\alpha)_p$ has the DP1 property if and only if each $X_\alpha$ does. This is not the case for $(\bigoplus_{\alpha \in \mathcal{A}} X_\alpha)_\infty$. In fact, we show that $(\bigoplus_{\alpha \in \mathcal{A}} X_\alpha)_\infty$ has the DP1 property if and only if it has the Dunford-Pettis property. A similar result also holds for vector-valued continuous function spaces.

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with \( \|x\| = (\sum_{\alpha \in \mathcal{A}} \|x_\alpha\|^p)^{1/p} \), and
\[
\left( \bigoplus_{\alpha \in \mathcal{A}} X_\alpha \right)_\infty = \left\{ x = (x_\alpha) \in \prod_{\alpha \in \mathcal{A}} X_\alpha : \sup_{\alpha \in \mathcal{A}} \|x_\alpha\| < \infty \right\},
\]
with \( \|x\| = \sup_{\alpha \in \mathcal{A}} \|x_\alpha\| \).

**Theorem 1.** Let \( X = (\bigoplus_{\alpha \in \mathcal{A}} X_\alpha)_\infty \). If \( Z \) is a Banach space, then a bounded linear operator \( T : X \to Z \) is DP1 if and only if it is completely continuous.

**Proof.** Suppose that \( T : X \to Z \) is DP1 but not completely continuous. Let \( \beta \in \mathcal{A}, Y_1 = X_\beta, \) and \( Y_2 = (\bigoplus_{\alpha \in \mathcal{A}} X_\alpha) \). Thus \( X = Y_1 \oplus Y_2 \). Since \( T \) is not completely continuous, there exists a normalized weakly null sequence \( (x_n,y_n) \) in \( Y_1 \oplus Y_2 \) and \( \varepsilon > 0 \) such that \( \|T((x_n,y_n))\| > \varepsilon \) for each \( n \in \mathbb{N} \). Define \( T_1 : Y_1 \to Z \) and \( T_2 : Y_2 \to Z \) by \( T_1(x) = T((x,0)) \) and \( T_2(y) = T((0,y)) \). It is clear that \( T((x,y)) = T_1(x) + T_2(y) \). Thus, by passing to a subsequence if needed, we may assume that either (a) \( \|T_1(x_n)\| > \varepsilon/2 \) or (b) \( \|T_2(y_n)\| > \varepsilon/2 \), for each \( n \in \mathbb{N} \). Without loss of generality, we assume that (a) holds. Let \( y \in Y_2 \) such that \( \|y\| = 1 \). As \( (x_n) \) is weakly null in \( X_1 \), the sequence \( ((x_n,y))^{\infty}_{n=1} \) converges weakly to \((0,y)\), and each \( (x_n,y) \) and \((0,y)\) belong to \( S_{X_1@X_2} \). As \( T \) is DP1, \( T((x_n,y)) \to T((0,y)) \) in norm. Therefore, \( T_1(x_n) \to 0 \) in norm, which is a contradiction.

**Corollary 2.** Let \( X = (\bigoplus_{\alpha \in \mathcal{A}} X_\alpha)_\infty \). Then \( X \) has the DP1 property if and only if it has the DPP. Likewise, \( X \) has the KKP if and only if \( X \) is a Schur space.
**Proof.** Suppose that \(m \rightarrow T : C(H, X) \rightarrow Y\) is a DP1 operator, but \(m\) is not strongly bounded. Then there exist sequences \((U_n)\) of pairwise disjoint open subsets of \(H\) and \((f_n)\) in \(C(H, X)\) such that \(\|f_n\| = 1\), \(\text{supp}(f_n) \subseteq U_n\), and \(\|T(f_n)\| > \epsilon\) for each \(n \in \mathbb{N}\) (see [1, Theorem 2.8]). Define \(g_n = f_1 + f_{n+1}\) for each \(n\). As \((f_n)\) is weakly null, \((g_n)\) converges weakly to \(f_1\) and \(\|g_n\| = 1\) for each \(n \in \mathbb{N}\).

Since \(T\) is DP1, \(T(g_n) \rightarrow T(f_1)\) in norm, and hence \(T(f_{n+1}) \rightarrow 0\) in norm. This is a contradiction.

**Theorem 4.** Let \(X\) and \(Y\) be Banach spaces, \(H\) a compact Hausdorff space containing at least two elements, and \(\Sigma\) the Borel subsets of \(H\). Then a bounded linear operator \(m \rightarrow T : C(H, X) \rightarrow Y\) is DP1 if and only if it is completely continuous. Thus \(C(H, X)\) has the DP1 property if and only if it has the DPP.

**Proof.** Suppose that \(m \rightarrow T : C(H, X) \rightarrow Y\) is DP1 but not completely continuous. Then there exists a normalized weakly null sequence \((f_n)\) in \(C(H, X)\) and \(\epsilon > 0\) such that \(\|T(f_n)\| > 6\epsilon\). Let \(U\) and \(V\) be nonempty open subsets of \(H\) such that \(U \cap V = \emptyset\). As \(T\) is DP1, Lemma 3 tells us that \(m\) is strongly bounded, and hence we may choose a regular nonnegative measure \(\mu\) defined on \(\Sigma\) such that \(\lim \mu(A) \to 0\) \(\hat{m}(A) = 0\).

For each \(n \in \mathbb{N}\), either (a) \(\|\int_{U \cup A} g_n dm\| > 3\epsilon\) or (b) \(\|\int_{H \setminus U} f_n dm\| > 3\epsilon\). By passing to a subsequence, we will assume that (a) holds for all \(n \in \mathbb{N}\). Let \(\varphi_1 : H \to [0, 1]\) be a continuous function such that \(\varphi(U) = 1\) and \(\varphi(V) = 0\). Next let \(\delta > 0\) be such that if \(\mu(A) < \delta\), then \(\hat{m}(A) < \epsilon\). Use Lusin’s theorem to obtain a continuous function \(\varphi_2 : H \to [0, 1]\) such that if \(A = \{x : \varphi_2(x) \neq \chi_U(x)\}\), then \(\mu(A) < \delta\). Let \(g \in C(H, X)\) such that \(\|g\| = 1\) and the support of \(g\) is contained in \(\overline{V}\). For each \(n \in \mathbb{N}\), define \(g_n = \varphi_1 \varphi_2 f_n\). This sequence is weakly null in \(C(H, X)\) as \((f_n)\) is. Thus \((g_n + g)\) converges weakly to \(g\). Also \(\|g_n + g\| = 1\) for each \(n \in \mathbb{N}\). As \(T\) is DP1, we have that \(\|T(g_n + g) - T(g)\| \to 0\), and thus \(\|T(g_n)\| \to 0\).

\[
\|T(g_n)\| = \left\| \int_{U \cup A} g_n dm \right\|
\geq \left\| \int_{U \cup A} g_n dm \right\| - \left\| \int_A g_n dm \right\|
\geq \left\| \int_{U \cup A} f_n dm \right\| - \hat{m}(A) \|g_n\|
\geq \left\| \int_{U \cup A} f_n dm \right\| - \hat{m}(A) (\|g_n\| + \|f_n\|)
\geq 3\epsilon - 2\epsilon = \epsilon,
\]

which yields the desired contradiction.

Now suppose that (b) holds. Use the fact that \(\mu\) is a regular “control” measure for \(\hat{m}\) to obtain a closed subset \(K \subseteq H \setminus \overline{U}\) such that \(\int_K f dm > 3\epsilon\). The proof follows as above by replacing, in the proof of case (a), \(\overline{U}\) and \(\overline{V}\) with \(K\) and \(\overline{U}\), respectively.
respectively, and again obtaining the desired contradiction. Hence $T$ must be completely continuous.

We conclude with the following observation. Since every DP1 operator on $(\bigoplus_{\alpha \in \mathcal{I}} X_{\alpha})_{\infty}$ and $C(H, E)$ are completely continuous, they are also unconditionally converging. Perhaps a reasonable question may be whether or not all DP1 operators are unconditionally converging. (Of course it is certainly the case that there are unconditionally converging operators that are not DP1.) To consider the converse notion, let $\|\cdot\|$ denote the norm on $c_0$, equivalent to the usual norm, defined by Day in [3]. It was shown in [7] that $(c_0, \|\cdot\|)$ is locally uniformly rotund, and hence has KKP. (See [8] for more details about $(c_0, \|\cdot\|)$.) It is straightforward to show that $X$ has the KKP if and only if for each $Y$, every bounded linear operator $T : X \to Y$ (in particular, any isomorphism of $X$) is DP1. Thus, using a classical result of Besagga and Pelczynski, we obtain the following theorem.

**Theorem 5.** If $X$ and $Y$ are Banach spaces, then a DP1 operator $T : X \to Y$ is not unconditionally converging if and only if $X$ contains a closed subspace $Z$ such that $Z$ is isometrically isomorphic to $(c_0, \|\cdot\|)$ and $T|_Z$ is an isomorphism.

**References**


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