THE CONJUGATION OPERATOR ON $A_q(G)$

SANJIV KUMAR GUPTA

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Let $q > 2$. We prove that the conjugation operator $H$ does not extend to a bounded operator on the space of integrable functions defined on any compact abelian group with the Fourier transform in $l_q$.

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Let $G$ be a compact abelian group with dual $\Gamma$. For $1 \leq q < \infty$, the space $A_q$ is defined as

$$A_q(G) = \{ f : f \in L^1(G), \hat{f} \in l_q(\Gamma) \}$$

with the norm $\|f\|_{A_q} = \|f\|_{L^1} + \|\hat{f}\|_{l_q}$. Then $A_q(G)$ is a commutative semisimple Banach algebra with maximal ideal space $\Gamma$, in which the set of trigonometric polynomials is dense [4]. The $A_p$-spaces have been studied in [1, 6].

If $G$ is, in addition, a connected group, then its dual can be ordered; there exists a semigroup $P \subset \Gamma$ such that $P \cap -P = \{0\}$, $P \cup -P = \Gamma$ (see [5]), and we say that $\gamma \in \Gamma$ is positive if $\gamma \in P$. If $f = \sum_{\gamma \in F} \hat{f}(\gamma) \gamma$ is a trigonometric polynomial, the conjugation operator $H$ is defined as

$$Hf = \sum_{\gamma \in F} \text{sgn}(\gamma) \hat{f}(\gamma) \gamma,$$

where $\text{sgn}(\gamma) = +1$ if $\gamma \in P$, $-1$ if $\gamma \in -P$, and $0$ if $\gamma = 0$.

If $1 \leq q \leq 2$, then $A_q(G) \subset L^2(G)$, and it is easy to see that $H$ extends to a bounded operator on $A_q(G)$. It was mentioned in [5] that the corresponding result for $q > 2$ is not known. Note that $H$ extends to a bounded operator on $A_q(G)$ if and only if $H$ extends to a bounded operator from $A_q(G)$ to $L^1(G)$. In [5], the following theorem was proved.

**Theorem 1.** Let $G$ be a compact, connected abelian group and $P$ any fixed order on $\Gamma$. If $q > 2$ and $\phi$ is a Young’s function, then the conjugation operator $H$ does not extend to a bounded operator from $A_q(G)$ to $L^\phi(G)$.

We prove in Theorem 2 that $H$ does not extend to a bounded operator on $A_q(G)$, $q > 2$, thus answering the problem mentioned in [5]. Also, Theorem 1 follows from our theorem (Theorem 2). Theorem 2 was proved for the circle group in [2] but for the completeness sake, we give it below.
**Theorem 2.** Let $G$ be a compact, connected abelian group and $P$ any fixed order on $\Gamma$. If $q > 2$, then the conjugation operator $H$ does not extend to a bounded operator on $A_q(G)$.

**Proof.** We will show that $H$ does not extend to a bounded operator from $A_q(G)$ to $L^1(G)$. The proof is divided into three steps.

**Step 1.** Let $G = T$, the circle group. Suppose that $H$ extends as a bounded operator from $A_q$ to $L^1$. Choose $\mu_0 \in M(T)$, $\hat{\mu}_0 \in l_q$ such that $\mu_0$ is not absolutely continuous. Define $T : L^1 \to L^1$ by

$$Tf = H(f \ast \mu_0),$$

where $T$ is well defined as $f \ast \mu_0 \in A_q$ and $H$ maps $A_q$ into $L^1$ by our assumption on $H$. Hence, $T$ is a multiplier from $L^1$ to $L^1$, and therefore is given by a measure $\mu \in M(T)$ (see [3]). Hence

$$\text{sgn}(n) \hat{\mu}_0(n) = \hat{\mu}(n).$$

Observe that

$$\hat{\mu}_0 = \frac{\hat{\mu}_0 + \hat{\mu}}{2} = \frac{\hat{\mu}_0 - \hat{\mu}}{2}.$$  

Now, $(\hat{\mu}_0 + \hat{\mu})/2$ and $(\hat{\mu}_0 - \hat{\mu})/2$ are absolutely continuous by F. and M. Riesz theorem. Hence, $\hat{\mu}_0$ is absolutely continuous, which contradicts the choice of $\mu_0$. Hence, $H$ is unbounded on $A_q$, $q > 2$.

**Step 2.** Let $I$ be a closed subgroup of $G$ such that $H$ does not extend as a bounded operator on $A_q(G/I)$. Then $H$ does not extend as a bounded operator on $A_q(G)$.

**Proof.** Let $(f_n)$ be a sequence of trigonometric polynomials on $G/I$ such that

$$\|Hf_n\|_{L^1(G/I)} \to \infty, \quad \|f_n\|_{A_q(G/I)} \to 0, \quad \text{as } n \to \infty. \quad (6)$$

Let $g_n = f_n \circ q$, where $q : G \to G/I$ is the quotient map. Then it is easily seen that $Hg_n = (Hf_n) \circ q$, $\|Hg_n\|_{L^1(G)} = \|Hf_n\|_{L^1(G/I)}$, and $\|f_n \circ q\|_{A_q(G)} = \|f_n\|_{A_q(G/I)}$. Hence

$$\|Hg_n\|_{L^1(G)} \to \infty, \quad \|g_n\|_{A_q(G)} \to 0, \quad \text{as } n \to \infty, \quad (7)$$

and Step 2 follows.

**Step 3.** Since $G$ is connected, $\Gamma$ contains an element of infinite order, say $\gamma_0$ (see [3]). Let $S$ denote the subgroup generated by $\gamma_0$ and let $I = S^\perp$. Then $G/H$ is isomorphic to the circle group $T$. Now, the proof of the theorem follows from Steps 1 and 2. 

$\square$
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REFERENCES


Sanjiv Kumar Gupta: Department of Mathematics and Statistics, College of Science, Sultan Qaboos University, P.O. Box 36, Al-Khaud 123, Sultanate of Oman

E-mail address: gupta@squ.edu.om