ON THE FRESNEL SINE INTEGRAL AND THE CONVOLUTION

ADEM KILIÇMAN

Received 19 November 2002 and in revised form 7 March 2003

The Fresnel sine integral $S(x)$, the Fresnel cosine integral $C(x)$, and the associated functions $S_+(x)$, $S_-(x)$, $C_+(x)$, and $C_-(x)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of the Fresnel sine integral and its associated functions with $x^r$, $x^r$ are evaluated.

2000 Mathematics Subject Classification: 33B10, 46F10.

1. Introduction. The Fresnel integrals occur in the diffraction theory and they are of two kinds: the Fresnel integral $S(x)$ with a sine in the integral and the Fresnel integral $C(x)$ with a cosine in the integral.

The *Fresnel sine integral* $S(x)$ is defined by

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin \frac{u}{2} \, du$$

(see [5]) and the associated functions $S_+(x)$ and $S_-(x)$ are defined by

$$S_+(x) = H(x)S(x), \quad S_-(x) = H(-x)S(x).$$

The *Fresnel cosine integral* $C(x)$ is defined by

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos \frac{u}{2} \, du$$

(see [5]) and the associated functions $C_+(x)$ and $C_-(x)$ are defined by

$$C_+(x) = H(x)C(x), \quad C_-(x) = H(-x)C(x),$$

where $H$ denotes Heaviside’s function.

We define the function $L_r(x)$ by

$$L_r(x) = \int_0^x u^r \sin \frac{u}{2} \, du$$
for \( r = 0, 1, 2, \ldots \). In particular, we have

\[
L_0(x) = \sqrt{\frac{\pi}{2}} S(x), \\
L_1(x) = \frac{1}{2} - \frac{1}{2} \cos x^2, \\
L_2(x) = \frac{1}{4} \sqrt{2} \sqrt{\pi} C(x) - \frac{1}{2} (\cos x^2) x.
\]

We define the functions \( \sin_+ x, \sin_- x, \cos_+ x, \) and \( \cos_- x \) by

\[
\begin{align*}
\sin_+ x &= H(x) \sin x, \\
\sin_- x &= H(-x) \sin x, \\
\cos_+ x &= H(x) \cos x, \\
\cos_- x &= H(-x) \cos x.
\end{align*}
\]

2. Convolution products. The classical definition for the convolution product of two functions \( f \) and \( g \) is as follows.

**Definition 2.1.** Let \( f \) and \( g \) be functions. Then the convolution \( f \ast g \) is defined by

\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t) \, dt
\]

for all points \( x \) for which the integral exists.

If the classical convolution \( f \ast g \) of two functions \( f \) and \( g \) exists, then \( g \ast f \) exists and

\[
f \ast g = g \ast f.
\]

Further, if \( (f \ast g)' \) and \( f \ast g' \) (or \( f' \ast g \)) exist, then

\[
(f \ast g)' = f \ast g' \quad \text{(or} f' \ast g \text{)}.
\]

The classical definition of the convolution can be extended to define the convolution \( f \ast g \) of two distributions \( f \) and \( g \) in \( \mathcal{D}' \) with the following definition, see [4].

**Definition 2.2.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \). Then the convolution \( f \ast g \) is defined by the equation

\[
\langle (f \ast g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x + y) \rangle \rangle
\]

for arbitrary \( \varphi \) in \( \mathcal{D}' \), provided that \( f \) and \( g \) satisfy either of the following conditions:

(a) either \( f \) or \( g \) has bounded support,

(b) the supports of \( f \) and \( g \) are bounded on the same side.
It follows that if the convolution $f \ast g$ exists by this definition, then (2.2) and (2.3) are satisfied.

**Theorem 2.3.** The convolution $(\sin_+ x^2) \ast x^r_+$ exists and

\[
(\sin_+ x^2) \ast x^r_+ = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i}(x) x^i_+ \tag{2.5}
\]

for $r = 0, 1, 2, \ldots$.

**Proof.** It is obvious that $(\sin_+ x^2) \ast x^r_+ = 0$ if $x < 0$. When $x > 0$, we have

\[
(\sin_+ x^2) \ast x^r_+ = \int_0^x \sin t^2 (x-t)^r dt
= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i}(x) x^i_+ \tag{2.6}
\]

thus proving (2.5). \[\square\]

**Corollary 2.4.** The convolution $(\sin_- x^2) \ast x^r_- \ast exists and

\[
(\sin_- x^2) \ast x^r_- = \sum_{i=0}^{r} \binom{r}{i} L_{r-i}(x) x^i_- \tag{2.7}
\]

for $r = 0, 1, 2, \ldots$.

**Proof.** Equation (2.7) follows on replacing $x$ by $-x$ in (2.5) and noting that

\[
L_r(-x) = (-1)^{r+1} L_r(x). \tag{2.8}
\]

\[\square\]

**Theorem 2.5.** The convolution $S_+(x) \ast x^r_+$ exists and

\[
S_+(x) \ast x^r_+ = \frac{\sqrt{2}}{\sqrt{\pi} (r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} L_{r-i+1}(x) x^i_+ \tag{2.9}
\]

for $r = 0, 1, 2, \ldots$.

**Proof.** It is obvious that $S_+(x) \ast x^r_+ = 0$ if $x < 0$. When $x > 0$, we have

\[
\sqrt{\frac{\pi}{2}} S_+(x) \ast x^r_+ = \int_0^x (x-t)^r \int_0^t \sin u^2 du \, dt
= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} L_{r-i+1}(x) x^i_+ \tag{2.10}
\]

Thus equation (2.9) follows. \[\square\]
**Corollary 2.6.** The convolution $S_-(x) \ast x_-'$ exists and

$$S_-(x) \ast x_-' = \frac{\sqrt{2}}{\sqrt{\pi} (r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} L_{r-i+1}(x)x_i^{i}$$

(2.11)

for $r = 0, 1, 2, \ldots$.

**Proof.** Equation (2.11) follows on replacing $x$ by $-x$ in (2.9).

3. Existence of neutrix convolution product. In order to extend the convolution product to a larger class of distributions, the neutrix convolution product was introduced in [1] and was later extended in [2, 3]. For the further extension, first of all, we let $\tau$ be a function in $\mathcal{S}$ having the following properties:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \leq \tau(x) \leq 1$,

(iii) $\tau(x) = 1$ for $|x| \leq 1/2$,

(iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function $\tau_\nu$ is now defined for $\nu > 0$ by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \leq \nu, \\
\tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\
\tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu. \end{cases}$$

(3.1)

**Definition 3.1.** Let $f$ and $g$ be distributions in $\mathcal{D}'$ and let $f_\nu = f \tau_\nu$ for $\nu > 0$. The neutrix convolution product $f \otimes g$ is defined as the neutrix limit of the sequence $\{f_\nu \ast g\}$, provided that the limit $h$ exists in the sense that

$$N_{-q} \lim \langle f_\nu \ast g, \varphi \rangle = \langle h, \varphi \rangle,$$

(3.2)

for all $\varphi$ in $\mathcal{D}$, where $N$ is the neutrix, see van der Corput [7], having domain $N'$, the positive real numbers, with negligible functions finite linear sums of the functions $\nu^\lambda \ln^{r-1} \nu$, $\ln^r \nu$, $\nu^r \sin \nu^2$, and $\nu^r \sin \nu^2$ ($\lambda \neq 0$, $r = 1, 2, \ldots$) and all functions which converge to zero in the normal sense as $\nu$ tends to infinity.

Note that in this definition the convolution product $f_\nu \ast g$ is defined in Gel’fand and Shilov’s sense, with the distribution $f_\nu$ having bounded support.

It was proved in [1] that if $f \ast g$ exists in the classical sense or by **Definition 2.1**, then $f \otimes g$ exists and

$$f \otimes g = f \ast g.$$ 

(3.3)

The following theorem was also proved in [1].

**Theorem 3.2.** Let $f$ and $g$ be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \otimes g$ exists. Then the neutrix convolution product $f \otimes g'$
exists and

\[(f \otimes g)' = f \otimes g'.\]  

(3.4)

Now if we let \(L_r = N^{-}\lim_{\nu \to \infty} L_r(\nu)\) and note that

\[S(\infty) = C(\infty) = \frac{1}{2},\]  

(3.5)

see Olver [6], then we have the following theorem.

**Theorem 3.3.** The neutrix convolution \((\sin x^2) \ast x^r\) exists and

\[(\sin x^2) \otimes x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i} x^i\]  

(3.6)

for \(r = 0, 1, 2, \ldots\)

**Proof.** We set

\[(\sin x^2)_\nu = (\sin x^2) \tau_\nu(x).\]  

(3.7)

Then the convolution \((\sin x^2)_\nu \ast x^r\) exists and

\[(\sin x^2)_\nu \ast x^r = \int_{0}^{\nu} \sin^2(x-t)r^r dt + \int_{\nu}^{\nu+\nu-\nu} \tau_\nu(t) \sin^2(x-t)r^r dt.\]  

(3.8)

Now

\[\int_{0}^{\nu} \sin^2(x-t)r^r dt = \sum_{i=0}^{r} \binom{r}{i} \int_{0}^{\nu} x^i(-t)^{r-i} \sin^2 dt = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i}(\nu) x^i,\]  

(3.9)

and it follows that

\[N^{-}\lim_{\nu \to \infty} \int_{0}^{\nu} \sin^2(x-t)r^r dt = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i}(\nu) x^i.\]  

(3.10)

Further, it can easily be seen that for each fixed \(x,\)

\[\lim_{\nu \to \infty} \int_{\nu}^{\nu+\nu-\nu} \tau_\nu(t) \sin^2(x-t)r^r dt = 0,\]  

(3.11)

and (3.6) follows from (3.9), (3.10), and (3.11).

**Theorem 3.4.** The neutrix convolution \(S_+(x) \otimes x^r\) exists and

\[S_+(x) \otimes x^r = \frac{\sqrt{2}}{\sqrt{\pi} (r+1)} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i+1} L_{r-i+1} x^i\]  

(3.12)

for \(r = 0, 1, 2, \ldots\)
**Proof.** We put $[S_+(x)]_v = S_+(x)\tau_v(x)$. Then the convolution product $[S_+(x)]_v \ast x^r$ exists and

$$[S_+(x)]_v \ast x^r = \int_0^v S(t)(x-t)^r dt + \int_v^{v+v-v} \tau_v(t)S(t)(x-t)^r dt. \quad (3.13)$$

We have

$$\sqrt{\frac{\pi}{2}} \int_0^v S(t)(x-t)^r dt = \int_0^v (x-t)^r \int_0^t \sin u^2 du dt \quad (3.14)$$

and it follows that

$$N^{-\lim}_{v \to \infty} \int_0^v S(t)(x-t)^r dt = \frac{\sqrt{\pi}}{\sqrt{\pi(r+1)}} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i+1} L_{r-i} x^i. \quad (3.15)$$

Further, it is easily seen that for each fixed $x$,

$$\lim_{v \to \infty} \int_v^{v+v-v} \tau_v(t)S(t)(x-t)^r dt = 0, \quad (3.16)$$

and (3.12) now follows immediately from (3.14), (3.15), and (3.16).

**Corollary 3.5.** The neutrix convolution $S_-(x) \odot x^r$ exists and

$$S_-(x) \odot x^r = \frac{\sqrt{\pi}}{\sqrt{\pi(r+1)}} \sum_{i=0}^r \binom{r+1}{i} (-1)^{r-i+1} L_{r-i+1} x^i \quad (3.17)$$

for $r = 0, 1, 2, \ldots$.

**Proof.** Equation (3.17) follows on replacing $x$ by $-x$ and $L_r$ by $(-1)^{r+1} L_r$ in (3.12).

**Corollary 3.6.** The neutrix convolution $S(x) \odot x^r$ exists and

$$S(x) \odot x^r = 0 \quad (3.18)$$

for $r = 0, 1, 2, \ldots$.

**Proof.** Equation (3.18) follows from (3.12) and (3.17) on noting that $S(x) = S_+(x) + S_-(x)$.

**Acknowledgments.** The author is very grateful to Professor H. M. Srivastava for his valuable comments and help in the improvement of this paper. The present research has been partially supported by Universiti Putra Malaysia under Grant IRPA 09-02-04-0259-EA001.
REFERENCES


Adem Kılıçman: Department of Mathematics and Institute of Advanced Technology (ITMA) and Mathematical Research Institute (INSPEM), University Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
E-mail address: akipic@fsas.upm.edu.my